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On the cyclic homology of exact categories

Bernhard Keller *

UFR de Mathématiques, URA 748 du CNRS, Université Paris 7, 2 Place Jussieu, 75251 Paris Cedex 05, France

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Abstract

The cyclic homology of an exact category was defined by McCarthy (1994) using the methods of Waldhausen (1985). McCarthy's theory enjoys a number of desirable properties, the most basic being the *agreement property*, i.e. the fact that when applied to the category of finitely generated projective modules over an algebra it specializes to the cyclic homology of the algebra.

However, we show that McCarthy's theory cannot be both compatible with localizations and invariant under functors inducing equivalences in the derived category.

This is our motivation for introducing a new theory for which all three properties hold: extension, invariance and localization. Thanks to these properties, the new theory can be computed explicitly for a number of categories of modules and sheaves. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

0.1. Overview of the results

Let k be a commutative ring and $\mathscr{A}$ an exact category in the sense of [30] which is moreover k-linear, i.e. the groups $\mathrm{Hom}_k(A,B)$, $A,B \in \mathscr{A}$, are endowed with k-module structures such that the composition is bilinear.

In [26], McCarthy has defined the Hochschild, cyclic, negative and periodic homologies of $\mathscr{A}$. He showed that they enjoy the following properties

* E-mail: keller@math.jussieu.fr.

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(1) *Agreement.* For the exact category of finitely generated projective modules over a (unital) algebra, the homologies agree with those of the algebra.

(2) *Exact sequences.* The different homologies are linked by the classical morphisms and long exact sequences.

(3) *Additivity.* The homologies are additive in the sense that the map induced by the middle term of a short exact sequence of functors is the sum of the maps induced by the outer terms.

(4) *Products.* The homologies admit product structures which agree with the classical structures in the situation of (1).

(5) *Trace maps.* There are trace maps linking the Quillen K-theory of $\mathscr{A}$ to its Hochschild resp. negative cyclic homology. Again these are compatible with the classical maps in the situation of (1),

Now by analogy with K-theory, there are two other properties which we might expect to hold for a homology theory of exact categories, namely

(6) *Invariance.* The theory should be preserved by exact functors inducing equivalences in the bounded derived categories [34, 35].

(7) *Localization.* It should be compatible with localizations (in a sense to be made precise).

These properties have been shown to hold for K-theory in many situations [32, 39, 38]. They have also been proved for cyclic homology of DG algebras in [19]. Unfortunately, the homologies defined by McCarthy cannot satisfy (1), (6), and (7). Indeed, we show in Examples 1.8 and 1.9 that a theory satisfying (1), (6), and (7) necessarily takes non-zero values in arbitrarily negative degrees, whereas the homologies defined by McCarthy are concentrated positive degrees by their definition. This also shows that McCarthy's cyclic homology cannot possibly satisfy the natural scheme-theoretic variant of property (1), which states that for the category $\mathrm{vec}\, X$ of vector bundles on a scheme X with an ample line bundle, there is a natural isomorphism

$$HC_*(\mathrm{vec}\, X) \xrightarrow{\sim} HC_*(X),$$

where $HC_*(X)$ is the cyclic homology of the scheme X as defined by Loday [22] and Weibel [37]. Indeed, $HC_*(X)$ contains the cohomology $H^*(X, \mathcal{O}_X)$ as a direct factor (concentrated in homologically negative degrees).

In this article, we propose a new definition of the Hochschild, cyclic ... homologies of an exact category and show that the new theories do satisfy (1)–(3), (6) and (7). Thanks to the two last properties, we are able to compute them for a number of non-trivial examples (finitely generated modules over noetherian algebras of finite global dimension in Section 1.6, coherent sheaves on projective space in Section 1.7, finite-length modules over $k[[X]]$ in Section 1.8, coherent sheaves on punctured affine space in Section 1.9, finitely generated modules over the dual numbers in Section 2.5). In many other cases, the computation may be reduced to that of the cyclic homology of a suitable differential graded algebra (Example 2.6). The new theories can also be shown to satisfy the scheme-theoretic analog of property (1), cf. Example 1.10.

We do not doubt that properties (4) and (5) also hold for the new theories. We provide evidence for this by proving a delooping theorem (Section 1.13) for the new theories in the case of a flat exact category (i.e. $\mathrm{Hom}_{\mathscr{A}}(A,B)$ is a flat k-module for all $A,B\in\mathscr{A}$). We can then construct a natural transformation

$$HC_*^{\mathrm{McC}}\mathscr{A}\to HC_*^{\mathrm{new}}\mathscr{A}$$

(and similarly for the other homologies) and define trace maps by composing this morphism with the trace maps constructed by McCarthy.

0.2. Organization of the article

In Section 1, we state the main results of the article and give some examples: We define the mixed complex of a flat exact category in Section 1.4. The homologies associated with the category are derived from its mixed complex. This ensures the validity of property (2). The main theorem (Section 1.5) states that properties (1), (6), and (7) hold. We illustrate the strength of (6) and (7) on some examples (Sections 1.6–1.8). In Section 1.12, the additivity property (3) is seen to be a consequence of the localization property (6). In turn, additivity is the essential ingredient for proving the delooping theorem in Section 1.13. Inspired by Kassel's work [14, 16] on bivariant theories we show in Section 1.14 that certain (non-exact) k-linear functors which admit total derived functors induce maps in the new theories, and that (6) and (7) continue to hold for this wider class of functors. In the last paragraph of Section 1, we prove a useful lemma which gives a sufficient condition for an exact sequence of abelian categories to induce an exact sequence of derived categories.

In Section 2, we restate the main theorem in a setting which contains both the results of Section 1 on exact categories and those of [19] on DG algebras as special cases. In this more general setting, categories of complexes or of DG modules are replaced by what we call 'exact DG categories' (Section 2.1). Each exact DG category has an associated triangulated category which generalizes the homotopy category of a category of complexes or of DG modules. An exact category gives actually rise to a pair of exact DG categories: the category of complexes and its full subcategory of acyclic subcomplexes. This pair is an example of a 'localization pair' (Section 2.4). Localization pairs are to be viewed as a more intrinsic variant of Thomason–Trobaugh's bicomplicial Waldhausen categories [32]. Each localization pair has associated with it a mixed complex and a triangulated category. In the case of the localization pair associated with an exact category, these are respectively the mixed complex of the exact category and its derived category. The analog of the main theorem holds for localization pairs (Section 2.4). This includes in particular the results of [19] for DG algebras as special cases. It improves on [19] in so far as we no longer have to make any hypothesis on the ground ring or on the underlying DG module of the algebra. This means, however, that we have to use a more elaborate definition of the mixed complex of a non-flat DG algebra using resolutions (Section 3.2).

As an application, we compute the mixed complex of the category of finite-dimensional modules over the dual numbers (Section 2.5). By the same method, we reduce the computation of the mixed complex of the category of finite-dimensional modules over a finite-dimensional algebra A to the computation of the mixed complex of an associated DG algebra (whose homology is the Ext-algebra of the simple A-modules).

Sections 3 and 4 contain the proof of the main theorem for localization pairs. They form the technical heart of the article. In Section 3, we prove existence and unicity up to homotopy of resolutions of exact DG categories. We start with the special case of DG algebras in Section 3.2. Here the only technical difficulty is that our DG algebras can have non-vanishing homology in positive *and* in negative degrees. The passage from DG algebras to exact DG categories in Section 3.6 then involves replacing an algebra by an 'algebra with several objects' and taking into account the exact structure. In fact, we do not only prove existence and unicity of resolutions but, more precisely, we show that if we quotient the category of exact DG categories by a suitable homotopy relation, then in the quotient, the multiplicative system of all functors inducing equivalences in the associated triangulated categories admits a calculus of right fractions. The corresponding localization is denoted by $\mathscr{M}^b$. It is equivalent to its full subcategory whose objects are the flat exact DG categories. Using this equivalence we extend the mixed complex functor from flat exact DG categories to all exact DG categories.

By the definition of $\mathscr{M}^b$, passing from an exact DG category to its associated triangulated category is a functor from $\mathscr{M}^b$ to the category of triangulated categories. We define a sequence of $\mathscr{M}^b$ to be exact iff the associated sequence of triangulated categories is exact.

In Section 4, we study the 'completion' functor $\mathscr{M}^b \to \mathscr{M}$. Its effect is to assign to each exact DG category a new, larger, exact DG category whose associated triangulated category admits 'arbitrary' coproducts. We deduce from the Neeman–Ravenel–Thomason–Trobaugh–Yao theorem (Section 4.12), that the completion functor preserves exactness (Section 4.2) and that it becomes an equivalence when restricted to the full subcategory of $\mathscr{M}^b$ whose objects are the exact DG categories whose associated triangulated categories are Karoubian (Section 4.1). The exactness of a sequence of $\mathscr{M}$ is of course defined by passing to the associated triangulated categories. Surprisingly enough, such sequences are actually exact in the pointed category $\mathscr{M}$, i.e. the first term is a monomorphism, the second an epimorphism and the two form a kernel–cokernel pair (Section 4.6). This allows us to associate with each localization pair a functorial exact sequence of $\mathscr{M}$ and in particular a 'quotient category' which depends functorially and exactly on the localization pair (Sections 4.7 and 4.8). The final step is now to prove that the mixed complex functor is a '∂-functor' on $\mathscr{M}$. This is done in Sections 4.9 and 4.13. The proof of Section 4.13 closely follows [19, Section 6] but corrects an error which occurred in [19, Lemma 5.2].

1. Cyclic homology of exact categories

1.1. Exact categories and categories of complexes

Let k be a commutative ring and $\mathcal{A}$ a k-linear category (i.e. an additive category whose morphism spaces are k-modules such that the composition is bilinear). Suppose that $\mathcal{A}$ is exact in the sense of [30]. We use the terminology of [7, Ch. 9]: admissible monomorphisms are called *inflations*, admissible epimorphisms – *deflations*, and admissible short exact sequences – *conflations*. A complex N over $\mathcal{A}$ is *acyclic in degree n* if d_N^{n-1} factors as

$$\begin{tikzcd} N^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr, "p^{n-1}"'] & & N^n \\ & Z^{n-1} \arrow[ur, "i^{n-1}"'] & \end{tikzcd}$$

where p^{n-1} is a cokernel for d^{n-2} and a deflation, and i^{n-1} is a kernel for d^n and an inflation. The complex N is *acyclic* if it is acyclic in each degree. We denote by $\mathcal{A}c^b\mathcal{A}$ the category of all acyclic complexes N which are bounded, i.e. we have $N^n = 0$ for all large $|n|$. We denote by $\mathcal{C}^b\mathcal{A}$ the category of all bounded complexes over $\mathcal{A}$.

If X and Y are two complexes over $\mathcal{A}$, we have a differential $\mathbf{Z}$-graded k-module $\mathcal{H}om_{\mathcal{A}}(X, Y)$ whose nth component consists of the homogeneous morphisms f of degree n of $\mathbf{Z}$-graded objects $(X^p) \to (Y^q)$ and whose differential is given by $d(f) = d_Y \circ f - (-1)^n f \circ d_X$, where f is of degree n. Note that for two composable morphisms

$$g \in \mathcal{H}om_{\mathcal{A}}(X, Y), \qquad f \in \mathcal{H}om_{\mathcal{A}}(Y, Z)^n,$$

we have the Leibniz rule

$$d(fg) = (df) \circ g + (-1)^n f \circ dg.$$

Thus any category of complexes over $\mathcal{A}$ may be viewed as a *differential graded category* in the sense of [13, 18].

The *homotopy category* $\mathcal{H}^b\mathcal{A}$ has the same objects as $\mathcal{C}^b\mathcal{A}$ and its morphisms $X \to Y$ are in bijection with the elements of

$$H^0 \mathcal{H}om_{\mathcal{A}}(X, Y).$$

It is a triangulated category. The full subcategory of the acyclic complexes forms a triangulated subcategory of $\mathcal{H}^b\mathcal{A}$. The *derived category* $\mathcal{D}^b\mathcal{A}$ is the localization of $\mathcal{H}^b\mathcal{A}$ with respect to the subcategory of acyclic complexes. Note that this makes sense although the subcategory of acyclic complexes may not be épaisse (cf. [28]). Indeed, if $\mathcal{T}$ is a triangulated category and $\mathcal{S} \subset \mathcal{T}$ a full triangulated subcategory (which need not be épaisse), then the localization $\mathcal{T}/\mathcal{S}$ exists and morphisms in the localization are given by a calculus of left or right fractions; the kernel of the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is the épaisse closure of $\mathcal{S}$.

1.2. Reminder on mixed complexes

We use Kassel's approach [14]. Recall that a mixed complex is a triple (C,b,B) such that

$$(C,b) = (\cdots \to C_p \to C_{p-1} \to \cdots)$$

is a complex of k-modules and $B: C \to C$ is a homogeneous morphism of $\mathbf{Z}$-graded k-modules of degree 1 satisfying $bB + Bb = 0$. Let Λ be the DG algebra generated by an indeterminate ε of chain degree 1 with $\varepsilon^2 = 0$ and $d\varepsilon = 0$. The underlying complex of Λ is

$$\cdots 0 \to k\varepsilon \xrightarrow{0} k \to 0 \cdots .$$

Then a mixed complex may be identified with a DG left Λ-module whose underlying DG k-module is (C,b) and where ε acts by B. This interpretation leads to the following definitions: Suppose that $C = (C,b,B)$ is a mixed complex. Then *the shifted mixed complex* $C[1]$ is the mixed complex such that $C[1]_p = C_{p-1}$ for all p, $b_{C[1]} = -b_C$, and $B_{C[1]} = -B_C$. Let $f: C \to C'$ be a morphism of mixed complexes. Then the *mapping cone over* f is the mixed complex

$$\left(C' \oplus C[1], \begin{bmatrix} b_{C'} & f \\ 0 & -b_C \end{bmatrix}, \begin{bmatrix} B_{C'} & 0 \\ 0 & -B_C \end{bmatrix} \right).$$

We define $\mathcal{M}ix$ to be the category of mixed complexes and $\mathcal{DM}ix$ to be the *mixed derived category*, i.e. the derived category of the DG algebra Λ. Its objects may be viewed as mixed complexes. Cyclic homology, Hochschild homology etc. may be interpreted as cohomological functors on $\mathcal{DM}ix$, cf. [19, 2.2]. Note that despite the notation, $\mathcal{DM}ix$ is *not* the derived category of the abelian category $\mathcal{M}ix$ (the objects of the derived category of $\mathcal{M}ix$ would be complexes of mixed complexes ...).

1.3. The mixed complex of a flat DG category

Let $\mathcal{B}$ be a small DG category, i.e. a category enriched in differential graded k-modules (cf., e.g. [18]). Assume that $\mathcal{B}$ is flat, i.e. $\mathcal{B}(A,B) \otimes_k N$ is acyclic for each acyclic DG k-module N and all $A, B \in \mathcal{B}$. In analogy with the construction of Hochschild–Mitchell homology [27] and with the case of DG algebras [9,33], we associate a precyclic chain complex with $\mathcal{B}$ as follows: For each $n \in \mathbf{N}$, its nth term is

$$\coprod \mathcal{B}(B_n, B_0) \otimes \mathcal{B}(B_{n-1}, B_n) \otimes \mathcal{B}(B_{n-2}, B_{n-1}) \otimes \cdots \otimes \mathcal{B}(B_0, B_1),$$

where the sum runs over all sequences $B_0, \ldots, B_n$ of objects of $\mathcal{B}$. The degeneracy maps are given by

$$d_i(f_n, \ldots, f_i, f_{i-1}, \ldots, f_0) = \begin{cases} (f_n, \ldots, f_i f_{i-1}, \ldots, f_0) & \text{if } i > 0, \\ (-1)^{(n+\sigma)}(f_0 f_n, \ldots, f_1) & \text{if } i = 0, \end{cases}$$

where $\sigma = (\deg f_0)(\deg f_1 + \cdots + \deg f_{n-1})$. The cyclic operator is given by

$$t_n(f_{n-1}, \ldots, f_0) = (-1)^{n+\sigma}(f_0, f_{n-1}, f_{n-2}, \ldots, f_1).$$

We associate a mixed complex $C(\mathscr{B})$ with this precyclic chain complex as described in [19, Section 2]. We view $C(\mathscr{B})$ as an object of the mixed derived category $\mathscr{D}\mathscr{M}ix$ as explained above (Section 1.2). By definition, the cyclic homology of $\mathscr{B}$ is the cyclic homology of the mixed complex $C(\mathscr{B})$, and similarly for the other variants of the theory (Hochschild, periodic, negative, ...). The cyclic complex of a DG category which is not necessarily flat will be defined via a flat resolution in Section 3.2.

1.4. The mixed complex associated with an exact category (flat case)

In the setting of (Section 1.1), suppose that $\mathscr{A}$ is small and *flat over* k, i.e. $\mathscr{A}(A,B)$ is a flat k-module for all $A, B \in \mathscr{A}$ (this holds, for example, if k is a field; an important non-example is the category of finitely generated abelian groups viewed as a $\mathbf{Z}$-linear category). Then the mixed complex associated with the exact category $\mathscr{A}$ is defined to be

$$C(\mathscr{A}) = \operatorname{Cone}(C(\mathscr{A}c^b\mathscr{A}) \to C(\mathscr{C}^b\mathscr{A})).$$

Here, $\mathscr{A}c^b\mathscr{A}$ and $\mathscr{C}^b\mathscr{A}$ are viewed as differential graded categories and C is the functor defined in (Section 1.3). Clearly $C(\mathscr{A})$ is functorial with respect to exact functors.

The definition of $C(\mathscr{A})$ for exact categories which are not necessarily flat over the ground ring is given in Section 3.9 using flat resolutions.

1.5. The main theorem

Let k be a commutative ring. All exact categories below are assumed to be k-linear and small. By an exact functor, we always mean a k-linear exact functor. Statements (b) and (c) below will be extended to certain non-exact functors in Section 1.14.

A *factor-dense* subcategory $\mathscr{A}'$ of an additive category $\mathscr{A}$ is a full subcategory such that each object of $\mathscr{A}$ is a direct factor of a finite direct sum of objects of $\mathscr{A}'$. An *equivalence up to factors* is an additive functor $\mathscr{A} \to \mathscr{B}$ which induces an equivalence onto a factor-dense subcategory of $\mathscr{B}$. A sequence

$$0 \to \mathscr{T}' \to \mathscr{T} \xrightarrow{Q} \mathscr{T}'' \to 0$$

of triangulated categories is *exact up to factors* if $\mathscr{T}'$ identifies with a factor-dense subcategory of the kernel of Q and Q induces an equivalence from $\mathscr{T}/\ker Q$ onto a factor-dense subcategory of $\mathscr{T}''$.

Theorem. (a) *If A is a k-algebra, there is a natural isomorphism in the mixed derived category*

$$C(A) \xrightarrow{\sim} C(\operatorname{proj} A).$$

(b) *If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between exact categories which induces an equivalence up to factors $\mathcal{D}^b\mathcal{A} \to \mathcal{D}^b\mathcal{B}$, then F induces an isomorphism in the mixed derived category*

$$C(\mathcal{A}) \xrightarrow{\sim} C(\mathcal{B}).$$

(c) *If $F : \mathcal{A}' \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{A}''$ are exact functors between exact categories such that the sequence*

$$0 \to \mathcal{D}^b\mathcal{A}' \to \mathcal{D}^b\mathcal{A} \to \mathcal{D}^b\mathcal{A}'' \to 0$$

is exact up to factors, then there is a canonical morphism $\partial(F,G)$ such that the sequence

$$C(\mathcal{A}') \to C(\mathcal{A}) \to C(\mathcal{A}'') \xrightarrow{\partial(F,G)} C(\mathcal{A}')[1]$$

is a triangle in the mixed derived category.

The theorem is a consequence of Section 2.4 below. Statement (b) is often applied in the following situation: Suppose that $\mathcal{A} \subset \mathcal{B}$ is a full subcategory closed under extensions. Consider the conditions

(a) For each $B \in \mathcal{B}$, there is an acyclic complex of $\mathcal{B}$

$$0 \to B \to A_0 \to \cdots \to A_n \to 0$$

with $A_i \in \mathcal{A}$ for all i.

(b) For each conflation $A \to B \to B'$ of $\mathcal{B}$ with $A \in \mathcal{A}$, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "1"'] & B \arrow[r] \arrow[d] & B' \arrow[d] \\ A \arrow[r] & A' \arrow[r] & A'' \end{tikzcd}$$

whose second row is a conflation of $\mathcal{A}$.

If (b) (or its dual) holds, then the inclusion induces a fully faithful functor $\mathcal{D}^b\mathcal{A} \to \mathcal{D}^b\mathcal{B}$. If moreover condition a) (or its dual) holds, then this functor is an equivalence (cf. also Section 1.15).

1.6. Example: algebras of finite global dimension

Let A be a noetherian algebra of finite global dimension over a commutative ring k. Let $\operatorname{mod} A$ denote the category of all finitely generated A-modules. I claim that the inclusion $\operatorname{proj} A \to \operatorname{mod} A$ induces an isomorphism

$$C(\operatorname{proj} A) \xrightarrow{\sim} C(\operatorname{mod} A)$$

in the mixed derived category. Indeed, the inclusion is an exact functor and the induced functor $\mathscr{D}^b\mathrm{proj}\,A \to \mathscr{D}^b\mathrm{mod}\,A$ is an equivalence by the above remark. So the claim follows from Section 1.5(b). If we combine it with Section 1.5(a), we find that if A is noetherian of finite global dimension, then we have a canonical isomorphism

$$C(A) \xrightarrow{\sim} C(\mathrm{mod}\,A)$$

in the mixed derived category.

1.7. Example: Projective space

Suppose that k is a field. Let n be a positive integer and V a vector space of dimension $n+1$ over k. Let A be the algebra of upper triangular matrices $(a_{ij})_{0\le i,j\le n}$ with $a_{ij} \in S^{j-i}V$ for $j \ge i$ and $a_{ij} = 0$ for $j < i$. Let $\mathbf{P}$ be the projectivization of V and $\mathrm{coh}\,\mathbf{P}$ the category of coherent sheaves on $\mathbf{P}$. There is a canonical fully faithful functor

$$\mathrm{proj}\,A \to \mathrm{coh}\,\mathbf{P}$$

taking the indecomposable projective right A-module $e_{ii}A$ to the sheaf $\mathscr{O}_{\mathbf{P}}(-i)$. By a theorem of Beilinson's [1], this functor induces an equivalence

$$\mathscr{D}^b\mathrm{proj}\,A \xrightarrow{\sim} \mathscr{D}^b\mathrm{coh}\,\mathbf{P}.$$

Thus by 1.5(b) and (a), we have isomorphisms

$$C(\mathrm{coh}\,\mathbf{P}) \xleftarrow{\sim} C(\mathrm{proj}\,A) \xleftarrow{\sim} C(A).$$

By [23, 1.2.15], we know that the inclusion of the diagonal matrices $D \subset A$ induces an isomorphism $C(D) \xrightarrow{\sim} C(A)$ in the mixed derived category. So we finally get the isomorphism

$$C(\mathrm{coh}\,\mathbf{P}) \xrightarrow{\sim} \bigoplus_{i=0}^{n} C(k).$$

1.8. Example: nilpotent matrices

Suppose that k is a field. Let $\mathscr{N}$ be the category of finite-length modules over the power series ring over k in one variable. An object of $\mathscr{N}$ is a finite-dimensional vector space endowed with a nilpotent endomorphism. The category $\mathscr{N}$ embeds into $\mathrm{mod}\,k[X]$ and its image equals the kernel of the localization functor $\mathrm{mod}\,k[X] \to \mathrm{mod}\,k[X,X^{-1}]$. At the level of derived categories, we obtain a short exact sequence

$$0 \to \mathscr{D}^b\mathscr{N} \to \mathscr{D}^b\mathrm{mod}\,k[X] \to \mathscr{D}^b\mathrm{mod}\,k[X,X^{-1}] \to 0.$$

Hence by 1.5(c), we have a triangle

$$C(\mathscr{N}) \to C(\mathrm{mod}\,k[X]) \to C(\mathrm{mod}\,k[X,X^{-1}]) \to C(\mathscr{N})[1]$$

in the mixed derived category. By example of Section 1.6, it is isomorphic to a triangle

$$C(\mathcal{N}) \to C(k[X]) \to C(k[X,X^{-1}]) \to C(\mathcal{N})[1].$$

Now if we take homology in degree 0, then the second morphism identifies with the injection $k[X] \to k[X,X^{-1}]$ so that we get

$$HC_{-1}\,\mathcal{N} = \operatorname{cok}\,(k[X] \to k[X,X^{-1}]).$$

To deduce this isomorphism, we have only used the statements in the theorem of Section 1.5. Thus, any homology theory satisfying the theorem in Section 1.5 must take non-zero values in negative degrees. The theories defined by McCarthy in [26] are concentrated in positive degrees by definition. So they cannot satisfy the theorem of Section 1.5.

1.9. Example: Punctured affine space

Let k be a field, $n \geq 2$ and $\mathcal{A}$ the category $\operatorname{coh} X$ of coherent sheaves on the scheme $X = \mathbf{A}^n \backslash \{0\}$. We will show that any homology theory satisfying the theorem of (Section 1.5) is non-zero in degree $-n+1$ when evaluated at the category $\mathcal{A}$.

This phenomenon admits the following simple explanation: The inclusion of the category of algebraic vectorbundles $\operatorname{vec} X$ into $\operatorname{coh} X$ yields an equivalence in the bounded derived categories. Now at least for the theory defined in Section 1.4, we know from theorem of Section 1.10 below that the cyclic homology of $\operatorname{vec} X$ is the cyclic homology of the scheme X in the sense of [22, 37]. The latter contains the cohomology of the scheme with coefficients in the structure sheaf as a direct factor and in the case of punctured affine space X we have $H^{n-1}(X, \mathcal{O}_X) \neq 0$.

To compute $HC_{-n+1}(\mathcal{A})$, we use the exact sequence of abelian categories

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{A} \to 0,$$

where $\mathcal{M}$ is the category of finitely generated modules over $A = k[X_1, \ldots, X_n]$ and $\mathcal{N}$ its full subcategory formed by the modules supported in $\{0\}$, i.e. such that all the X_i act nilpotently. Using example (b) of Section 1.15, we see that the functor $\mathcal{D}^b\mathcal{N} \to \mathcal{D}^b\mathcal{M}$ is fully faithful. Thus we obtain an exact sequence of derived categories

$$0 \to \mathcal{D}^b\mathcal{N} \to \mathcal{D}^b\mathcal{M} \to \mathcal{D}^b\mathcal{A} \to 0.$$

and an exact sequence

$$HC_{-n+1}(\mathcal{M}) \to HC_{-n+1}(\mathcal{A}) \to HC_{-n}(\mathcal{N}) \to HC_{-n}(\mathcal{M})$$

by theorem 1.5(c). We know from 1.5(a) and the example of Section 1.6 that $HC_i(\mathcal{M})$ vanishes for all $i < 0$. Thus we have an isomorphism

$$HC_{-n+1}(\mathcal{A}) \xrightarrow{\sim} HC_{-n}(\mathcal{N}).$$

To compute $HC_{-n}(\mathcal{N})$, we need to introduce some notation: For two subsets I, J of the set $\{1, \ldots, n\}$ let $\mathcal{M}_I^J$ denote the category of modules over $A = k[X_1, \ldots, X_n]$ where the X_i, $i \in I$ act nilpotently and the X_j, $j \in J$, invertibly, and which are finitely generated as modules over $A[X_j^{-1} \mid j \in J]$. Note that if $I \cap J \neq \emptyset$, then $\mathcal{M}_I^J$ is the zero category. If I or J is empty, we omit the corresponding symbol from the notation. Thus $\mathcal{M}$ is the category of finitely generated A-modules as defined before, and $\mathcal{N} = \mathcal{M}_{\{1,\dots,n\}}$.

We will show by induction on $r = |I|$ that $HC_p(\mathcal{M}_I^J) = 0$ for $p < -r$ and that

$$HC_{-r}(\mathcal{M}_I^J) = A_I^J,$$

where we define

$$A_I^J = A[X_k^{-1} \mid k \in I \cup J] \Big/ \sum_{i \in I} A[X_k^{-1} \mid k \in I \cup J \setminus \{i\}].$$

In particular, we have

$$HC_{-n}\,\mathcal{M} = A[X_1^{-1}, \dots, X_n^{-1}] \Big/ \sum_{i=1}^{n} A[X_1^{-1}, \dots, \widehat{X_i^{-1}}, \dots X_n^{-1}].$$

To start with the induction, note that we have $HC_0(\mathcal{M}^J) = A[X_j^{-1} \mid j \in J]$ and $HC_p(\mathcal{M}^J) = 0$ for $p < 0$ by 1.5(a) and the example of Section 1.6. For the step from r to $r+1$ let $j_{r+1} \notin J$ and put $J^+ = J \cup \{j_{r+1}\}$, $I^+ = I \cup \{j_{r+1}\}$. Consider the exact sequence of abelian categories

$$0 \to \mathcal{M}_{I^+}^J \to \mathcal{M}_I^J \to \mathcal{M}_I^{J^+} \to 0.$$

By example (b) of 1.15, this sequence induces an exact sequence in the derived categories. So from 1.5(c) we get the exact sequence

$$HC_{-r}(\mathcal{M}_I^J) \to HC_{-r}(\mathcal{M}_I^{J^+}) \to HC_{-r-1}(\mathcal{M}_{I^+}^J) \to HC_{-r-1}(\mathcal{M}_I^J).$$

Using the induction hypothesis we see that this sequence is isomorphic to

$$A_I^J \to A_I^{J^+} \to HC_{-r-1}(\mathcal{M}_{I^+}^J) \to 0.$$

It follows that $HC_{-r-1}(\mathcal{M}_{I^+}^J)$ identifies with $A_I^{J^+}/A_I^J \xrightarrow{\sim} A_{I^+}^J$.

1.10. Cyclic homology of schemes

Let X be a scheme over a field k which admits an ample line bundle (for example a quasi-projective variety). Let $\mathrm{vec}(X)$ denote the category of locally finitely generated free sheaves on X (i.e. the category of algebraic vector bundles). It is an exact subcategory of the category of quasi-coherent sheaves on X.

Theorem (Keller [20]). *There is a canonical isomorphism*

$$HC_*(X) \xrightarrow{\sim} HC_*(\mathrm{vec}(X)).$$

Here $HC_*(X)$ denotes the cyclic homology of the scheme X as defined by Loday [22] and Weibel [37]. For an affine scheme $X = \mathrm{Spec}(A)$, the category $\mathrm{vec}(X)$ is equivalent to $\operatorname{proj} A$ and on the other hand, Weibel has shown in [loc. cit.] that $HC_*(X)$ is canonically isomorphic to $HC_*(A)$. So the theorem reduces to the result 1.5(a) in this case.

It will be shown in [20] that the theorem above generalizes to arbitrary quasi-compact quasi-separated schemes if we replace the exact category $\operatorname{vec}(X)$ by the localization pair (Section 2.4) associated with the category of perfect complexes on X.

1.11. A counterexample to devissage

Suppose that k is a field. Let $A = k[\varepsilon]/(\varepsilon^2)$. In Section 2.5 below we will show that $HH_*(\operatorname{mod} A) = A \otimes_k k[T]$ where T is of homological degree -1. As a graded k-module, this is clearly non isomorphic to $HH_*(\operatorname{mod} k) = k[u]$, where u is of homological degree 2. This example shows that the analogue of the dévissage theorem 4 of [30] does not hold for the invariant $A \mapsto HH_*(\operatorname{mod} A)$.

1.12. Application: Additivity

Let k be a commutative ring and $\mathcal{A}$ a small k-linear exact category. Let $\operatorname{con} \mathcal{A}$ denote the category of conflations

$$\varepsilon : A \xrightarrow{i} B \xrightarrow{d} C$$

of $\mathcal{A}$. It becomes an exact category if we endow it with the componentwise conflations. Let $P : \operatorname{con} \mathcal{A} \to \mathcal{A}$ be the functor $\varepsilon \mapsto C$ and $R : \operatorname{con} \mathcal{A} \to \mathcal{A}$ the functor $\varepsilon \mapsto A$.

Theorem. *The functors P and R induce an isomorphism in the mixed derived category*

$$C(\operatorname{con} \mathcal{A}) \xrightarrow{\sim} C(\mathcal{A}) \oplus C(\mathcal{A}).$$

Proof. Let $I : \mathcal{A} \to \operatorname{con} \mathcal{A}$ denote the functor

$$A \mapsto (A \xrightarrow{\mathbf{1}_A} A \to 0).$$

Then we have a short exact sequence of derived categories

$$0 \to \mathcal{D}^b \mathcal{A} \xrightarrow{I} \mathcal{D}^b \operatorname{con} A \xrightarrow{P} \mathcal{D}^b \mathcal{A} \to 0.$$

Hence by Section 1.5(c), we have a triangle

$$C(\mathcal{A}) \xrightarrow{C(I)} C(\operatorname{con} \mathcal{A}) \xrightarrow{C(P)} C(\mathcal{A}) \to C(\mathcal{A})[1]$$

in $\mathcal{DM}ix$. Now the functor R satisfies $RI = \mathbf{1}_{\mathcal{A}}$. So the triangle splits and we get the isomorphism of the theorem. $\square$

1.13. Application: Delooping

Let k be a commutative ring and $\mathscr{A}$ a small k-linear flat exact category. Recall from Section 1.4 that

$$C(\mathscr{A}) = \mathrm{Cone}\,(C(\mathscr{A}c^b\mathscr{A}) \to C(\mathscr{C}^b\mathscr{A})).$$

This definition clearly defines a functor from the category of small exact flat categories and exact functors to the category of mixed complexes. Hence it admits a natural extension to a functor from simplicial exact categories to simplicial mixed complexes. In particular, if $S_\bullet\mathscr{A}$ denotes the Waldhausen construction (see [36; 26, 3.1]), we have a simplicial object of mixed complexes $C(S_\bullet\mathscr{A})$. We denote by $\mathrm{Tot}\, C(S_\bullet\mathscr{A})$ the mixed complex obtained by passing from the simplicial object $C(S_\bullet\mathscr{A})$ to its associated reduced chain complex (whose components are mixed complexes) and then to the (sum) total mixed complex.

Theorem. *There is a canonical isomorphism in the mixed derived category*

$$\mathrm{Tot}\, C(S_\bullet\mathscr{A}) \xrightarrow{\sim} C(\mathscr{A})[1].$$

Proof. Consider the sequence

$$0 \to const_\bullet\mathscr{A} \to P_\bullet S_\bullet\mathscr{A} \to S_\bullet\mathscr{A} \to 0$$

of simplicial exact categories: here $const_\bullet\mathscr{A}$ denotes the constant simplicial category with value $\mathscr{A}$ and $P_\bullet S_\bullet\mathscr{A}$ is the 'path object' of $S_\bullet\mathscr{A}$ (see [26, 3.3]). Let $\mathscr{I}_n\mathscr{A}$ denote the category whose objects are the sequences

$$A_0 \to A_1 \to \cdots \to A_n$$

of inflations of $\mathscr{A}$. The nth component of the above sequence is given (up to equivalence) by

$$0 \to \mathscr{A} \xrightarrow{F} \mathscr{I}_n\mathscr{A} \xrightarrow{G} \mathscr{I}_{n-1}\mathscr{A} \to 0,$$

where F maps $A \in \mathscr{A}$ to the constant sequence

$$A \xrightarrow{1} A \xrightarrow{1} \cdots \xrightarrow{1} A$$

and G maps a sequence $A_0 \to \cdots \to A_n$ to

$$A_1/A_0 \to \cdots \to A_n/A_0.$$

This is clearly a (split) exact sequence of exact categories. By the additivity (Section 1.12), this implies that in

$$C(const_n\mathscr{A}) \to C(P_nS_\bullet\mathscr{A}) \to C(S_n\mathscr{A})$$

the first morphism is a quasi-isomorphism onto the kernel of the second, which is surjective in each component. Now if $\mathscr{B}_\bullet$ is a simplicial exact category, then $\mathrm{Tot}\, C(\mathscr{B}_\bullet)$

is filtered by a complete, bounded below filtration with subquotients the $C(\mathscr{B}_n)$ suitably shifted. This implies that the morphism

$$\operatorname{Tot} C(const_\bullet \mathscr{A}) \to \operatorname{Tot} C(P_\bullet S_\bullet \mathscr{A})$$

is a quasi-isomorphism onto the kernel of the morphism

$$\operatorname{Tot} C(P_\bullet S_\bullet \mathscr{A}) \to \operatorname{Tot} C(S_\bullet \mathscr{A}),$$

which is surjective in each component. Hence we have a canonical triangle in the mixed derived category

$$\operatorname{Tot} C(const_\bullet \mathscr{A}) \to \operatorname{Tot} C(P_\bullet S_\bullet \mathscr{A}) \to \operatorname{Tot} C(S_\bullet \mathscr{A}) \to \operatorname{Tot} C(const_\bullet \mathscr{A})[1].$$

Now we have a canonical isomorphism of mixed complexes $\operatorname{Tot} C(const_\bullet \mathscr{A}) \xrightarrow{\sim} C(\mathscr{A})$. Moreover $P_\bullet S_\bullet \mathscr{A}$ is contractible as a simplicial object, so $\operatorname{Tot} C(P_\bullet S_\bullet \mathscr{A})$ is a zero object in $\mathscr{D}\mathscr{M}ix$. So we have a canonical isomorphism $\operatorname{Tot} C(S_\bullet \mathscr{A}) \to C(\mathscr{A})[1]$ in $\mathscr{D}\mathscr{M}ix$. □

1.14. *Extended functoriality*

Let k be a commutative ring, $\mathscr{A}$ and $\mathscr{B}$ small k-linear exact categories and $F : \mathscr{A} \to \mathscr{B}$ a k-linear functor which is *not* necessarily exact. Inspired by Kassel's work [14, 16] we would like to assign to F a morphism $C(\mathscr{A}) \to C(\mathscr{B})$ of the mixed derived category. For this, we assume that the functor F is *right derivable*, i.e. that $\mathscr{A}$ admits a full exact subcategory $\mathscr{A}' \subset \mathscr{A}$ satisfying conditions (a) and (b) of the remark following Theorem 1.5 and such that the restriction of F to $\mathscr{A}'$ is exact. Then the total derived functor $\mathbf{R}F : \mathscr{D}^b\mathscr{A} \to \mathscr{D}^b\mathscr{B}$ exists in the sense of [4] and we have a diagram (commutative up to isomorphism)

$$\begin{tikzcd} \mathscr{D}^b\mathscr{A}' \arrow[r, "F \mid \mathscr{A}'"] \arrow[d, "\mathbf{1}"'] & \mathscr{D}^b\mathscr{B} \\ \mathscr{D}^b\mathscr{A}' \arrow[r, "\sim"] & \mathscr{D}^b\mathscr{A}. \arrow[u, "\mathbf{R}F"'] \end{tikzcd}$$

Accordingly, we *define the morphism* $C(\mathbf{R}F)$ in the mixed derived category by the commutative diagram

$$\begin{tikzcd} C(\mathscr{A}') \arrow[r, "C(F \mid \mathscr{A}')"] \arrow[d, "\mathbf{1}"'] & C(\mathscr{B}) \\ C(\mathscr{A}') \arrow[r, "\sim"] & C(\mathscr{A}). \arrow[u, "C(\mathbf{R}F)"'] \end{tikzcd}$$

We see that if $\mathbf{R}F$ is an equivalence up to factors, then $C(\mathbf{R}F)$ is an isomorphism by 1.5(b). Similarly, one can define $C(\mathbf{L}F)$ if F is left derivable (left to the reader).

In 1.5(c), instead of supposing that F and G are exact, it is enough to assume that they are right derivable and that the sequence

$$0 \to \mathscr{D}^b\mathscr{A}' \xrightarrow{\mathbf{R}F} \mathscr{D}^b\mathscr{A} \xrightarrow{\mathbf{R}G} \mathscr{D}^b\mathscr{A}'' \to 0.$$

is exact up to factors (similarly with 'right' replaced by 'left' for F, G, or both). Then we still have a canonical triangle

$$C(\mathscr{A}') \xrightarrow{C(\mathbf{R}F)} C(\mathscr{A}) \xrightarrow{C(\mathbf{R}G)} C(\mathscr{A}'') \xrightarrow{\partial(\mathbf{R}F,\mathbf{R}G)} C(\mathscr{A}')[1].$$

This is not a consequence of 1.5(c) but of its proof: Indeed, in the proof, we pass from an exact category to the corresponding localization pair (Section 2.4), to objects of the categories $\mathscr{L}^b$ and $\mathscr{L}$, and finally to objects of the category $\mathscr{M}$ (Section 4.8). For an exact category $\mathscr{A}$, denote by $M(\mathscr{A})$ the corresponding object of $\mathscr{M}$. Each object M of $\mathscr{M}$ has an associated triangulated category $\mathscr{T}M$ and $\mathscr{T}M(\mathscr{A})$ is a triangulated category whose subcategory of compact objects is equivalent up to factors to $\mathscr{D}^b\mathscr{A}$ (Section 4.8). The object $M(\mathscr{A})$ is functorial in $\mathscr{A}$ and an exact functor $\mathscr{A}' \to \mathscr{A}$ inducing an equivalence up to factors in the derived categories induces an isomorphism $M(\mathscr{A}') \to M(\mathscr{A})$ (Section 4.8). It follows that we can define $M(\mathbf{R}F)$ for a right derivable functor F in the natural way. Moreover, it follows from Section 4.8 that under the above hypotheses, the sequence

$$0 \to M(\mathscr{A}') \xrightarrow{M(\mathbf{R}F)} M(\mathscr{A}) \xrightarrow{M(\mathbf{R}G)} M(\mathscr{A}'') \to 0$$

is an exact sequence of $\mathscr{M}$ (Section 4.6). Now the assertion follows from Section 4.9 (b).

1.15. Localization: Abelian vs. derived categories

We refer to [6] for an introduction to the localization theory of abelian categories.

Let $\mathscr{B}$ be an abelian category and $\mathscr{A}$ a Serre subcategory. By definition, we have an exact sequence of abelian categories

$$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A} \to 0.$$

Such a sequence may or may not induce an exact sequence of derived categories, as we will see in the (non-) examples below. A sufficient condition is given in the following lemma. Even if the induced sequence in the derived categories is not exact, statement (a) of the lemma shows that we have an exact sequence of localization pairs, which still yields information on cyclic homology by Theorem 2.4 below.

Lemma. (a) *We have an exact sequence of triangulated categories*

$$0 \to \mathscr{D}^b_{\mathscr{B}}\mathscr{A} \to \mathscr{D}^b\mathscr{B} \to \mathscr{D}^b(\mathscr{B}/\mathscr{A}) \to 0,$$

where $\mathscr{D}^b_{\mathscr{A}}\mathscr{B}$ denotes the full subcategory of complexes whose homology lies in $\mathscr{A}$.

(b) *We have an exact sequence of derived categories*

$$0 \to \mathcal{D}^b\mathcal{A} \to \mathcal{D}^b\mathcal{B} \to \mathcal{D}^b(\mathcal{B}/\mathcal{A}) \to 0$$

iff the canonical functor $\mathcal{D}^b\mathcal{A} \to \mathcal{D}^b_{\mathcal{A}}\mathcal{B}$ *is an equivalence and this holds iff it is fully faithful.*

(c) *The condition of* (b) *holds and the canonical functor* $\mathcal{D}^b\mathcal{A} \to \mathcal{D}^b_{\mathcal{A}}\mathcal{B}$ *is an equivalence in each of the following cases*:

(c1) *For each exact sequence* $0 \to A \to B \to C \to 0$ *of* $\mathcal{A}$ *with* $A \in \mathcal{A}$, *there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0
\end{array}$$

where A' *and* A'' *belong to* $\mathcal{A}$.

(c2) *The abelian category* $\mathcal{A}$ *is generated by objects* X *of projective dimension at most* 1 (*i.e. we have* $\mathrm{Ext}^i_{\mathcal{A}}(X,?)=0$ *for all* $i \geq 2$).

Proof. (a) The canonical functor $\mathcal{C}^b(\mathcal{B})/\mathcal{C}^b(\mathcal{A}) \to \mathcal{C}^b(\mathcal{B}/\mathcal{A})$ is easily seen to be an equivalence. Its quasi-inverse induces a quasi-inverse to the canonical functor $\mathcal{D}^b(\mathcal{B})/\mathcal{D}^b_{\mathcal{A}}(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B}/\mathcal{A})$.

(b) By (a), the canonical functor is an equivalence iff the sequence is exact. If the functor is fully faithful, then it is an equivalence by devissage.

(c1) is well known [10].

(c2) For $i \leq 1$ and $L, M \in \mathcal{A}$, we have isomorphisms

$$\mathrm{Ext}^i_{\mathcal{A}}(L,M) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{B}}(L,M)$$

since $\mathcal{A}$ is a Serre subcategory. Now fix $M \in \mathcal{A}$ and, for $L \in \mathcal{A}$, put $E^i_{\mathcal{A}}L = \mathrm{Ext}^i_{\mathcal{A}}(L,M)$ and $E^i_{\mathcal{B}}L = \mathrm{Ext}^i_{\mathcal{B}}(L,M)$. The canonical map

$$E^i_{\mathcal{A}}L \to E^i_{\mathcal{B}}L$$

is clearly a morphism of δ-functors and it is invertible for $i = 0, 1$. To show that it is invertible for all $i \in \mathbf{N}$, it is enough to show that the functor $E^i_{\mathcal{B}} : \mathcal{A} \to \mathrm{Mod}\,\mathbf{Z}$ is effaceable for $i \geq 2$. This is immediate from the assumption. Thus the canonical map

$$\mathrm{Ext}^i_{\mathcal{A}}(L,M) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{B}}(L,M)$$

is an isomorphism for all $L, M \in \mathcal{A}$ and all $i \in \mathbf{N}$. By devissage, this implies that the canonical functor

$$\mathcal{D}^b\mathcal{A} \to \mathcal{D}^b_{\mathcal{A}}\mathcal{B}$$

is fully faithful. $\square$

Example. (a) *Localization of non-commutative rings.* Let B be a right coherent algebra and $S \subset B$ a subset such that

(a) $1 \in S$ and $SS \subset S$,

(b) For $s \in S$ and $b \in B$, there are $t \in S$ and $c \in B$ such that $cs = tb$.

(c) For $t \in S$ and $c \in B$, there are $s \in S$ and $b \in B$ such that $cs = tb$.

(d) For each $s \in S$, left multiplication by s is injective.

That is to say that S is a multiplicative subset (a) satisfying both Ore conditions (b), (c) and consisting of left non-zero divisors (d), cf. [5]. The ring of fractions $B[S^{-1}]$ is again right coherent. Let $\mathscr{B} = \operatorname{mod} B$ denote the category of finitely presented right B-modules and $\mathscr{A}$ the kernel of the localization functor $\operatorname{mod} B \to \operatorname{mod} B[S^{-1}]$. It is well-known (and easy to check) that the canonical functor $\mathscr{B}/\mathscr{A} \to \operatorname{mod} B[S^{-1}]$ is an equivalence. Thus we have an exact sequence of abelian categories

$$0 \to \mathscr{A} \to \operatorname{mod} B \to \operatorname{mod} B[S^{-1}] \to 0.$$

The category $\mathscr{A}$ is generated by the B/sB, $s \in S$, which are clearly of projective dimension at most 1 in $\operatorname{mod} B$. So by the lemma, we have an exact sequence of derived categories

$$0 \to \mathscr{D}^b \mathscr{A} \to \mathscr{D}^b \operatorname{mod} B \to \mathscr{D}^b \operatorname{mod} B[S^{-1}] \to 0.$$

(b) *Modules supported on a closed affine subscheme.* Let B be a commutative noetherian ring and $I \subset B$ an ideal. Let $\mathscr{B} = \operatorname{mod} B$ be the category of finitely generated B-modules and $\mathscr{A}$ its full subcategory consisting of the modules annihilated by some power of I. Let $S = B \setminus I$. Then we have an exact sequence of abelian categories

$$0 \to \mathscr{A} \to \operatorname{mod} B \to \operatorname{mod} B[S^{-1}] \to 0$$

and it does induce an exact sequence of derived categories. Indeed, we can use condition (c1): Let $0 \to N \to M \to L \to 0$ be an exact sequence of B-modules with $N \in \mathscr{A}$. Then by the Artin–Rees lemma [25, Theorem 8.5], there is an integer $c \geq 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$ for all $n > c$. So if we choose n such that $I^{n-c} N = 0$, we have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "1"'] & M \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & M/I^n M \arrow[r] & M/(I^n M + N) \arrow[r] & 0
\end{tikzcd}$$

whose second row belongs to $\mathscr{A}$.

(c) *A non-example.* Let k be a field and B the algebra of upper triangular 3×3 matrices over k divided by the ideal generated by the matrix E_{13}. Denote by S_i the simple right B-module given by the character sending $\sum a_{jk} E_{jk}$ to a_{ii}. Let $\mathscr{B}$ be the category of all finitely generated A-modules and $\mathscr{A}$ the full subcategory whose objects are the finite direct sums of copies of S_1 and S_3. We have $\operatorname{Ext}^1_B(S_1, S_3) = 0 = \operatorname{Ext}^1_B(S_3, S_1)$. So

$\mathscr{A}$ is closed under extensions. Clearly it is closed under subobjects and quotients. Thus it is a Serre subcategory. As an abelian category, $\mathscr{A}$ is semisimple and in particular $\mathrm{Ext}^2_{\mathscr{A}}(S_1,S_3)=0$. On the other hand, an easy computation shows that $\mathrm{Ext}^2_{B}(S_1,S_3)=k$. Hence the canonical functor $\mathscr{D}^b\mathscr{A}\to\mathscr{D}^b_{\mathscr{A}}\mathscr{B}$ is not an equivalence in this case.

2. Exact DG categories

2.1. *Definitions*

Let k be a commutative ring and $\mathscr{B}$ a small DG k-category (cf. [18]).

For example, let $\mathscr{A}$ be a small k-linear category. Then the category $\mathscr{B}=\mathscr{C}\mathscr{A}$ of chain complexes

$$A=(\cdots A^p \xrightarrow{d} A^{p+1}\to\cdots),\quad p\in\mathbf{Z},\ d^2=0,$$

over $\mathscr{A}$ becomes a DG k-category if we take $\mathscr{B}(A,B)$ to be the morphism complex (the nth component of $\mathscr{B}(A,B)$ is formed by the families $f=(f^p)$ of morphisms $A^p\to B^{n+p}$ and the differential is given by $d(f)=d\circ f-(-1)^n f\circ d)$.

Recall that a DG (right) $\mathscr{B}$-module is a DG functor $M:\mathscr{B}^{op}\to\mathrm{Dif}\,k$, where $\mathrm{Dif}\,k$ denotes the category of differential graded k-modules. The module M is given by differential graded k-modules $M(B)$, $B\in\mathscr{B}$, and morphisms of chain complexes

$$\mathscr{B}(B,C)\to(\mathrm{Dif}\,k)(M(C),M(B)),\quad b\mapsto M(b)$$

such that $M(b)M(a)=(-1)^{pq}M(ab)$ for $a\in\mathscr{B}(B,C)^q$ and $b\in\mathscr{B}(A,B)^p$. For a DG module M, we denote by $M[1]$ the shifted module: By definition, we have $M[1](B)^p=M(B)^{p+1}$ and $d_{M[1](B)}=-d_{M(B)}$ for all $B\in\mathscr{B}$, $p\in\mathbf{Z}$; moreover, for $b\in\mathscr{B}(B,C)^p$ we have $M[1](b)^q=(-1)^{pq}M(b)^{q+1}$. A *morphism of graded $\mathscr{B}$-modules* $f:M\to N$ is the datum of a morphism of $\mathbf{Z}$-graded k-modules $f(B):M(B)\to N(B)$ for each $B\in\mathscr{B}$ such that we have $f(B)M(b)=N(b)f(C)$ for each $b\in\mathscr{B}(B,C)$. A *morphism of differential graded $\mathscr{B}$-modules* is a morphism f of graded $\mathscr{B}$-modules such that $f(B)$ commutes with the differential for each $B\in\mathscr{B}$. If $f:M\to N$ is a morphism of DG modules, the *mapping cone* $\mathrm{Cone}(f)$ is the DG module K defined by $K(f)(B)^p=N(B)^p\oplus M(B)^{p+1}$,

$$d^p_{K(f)}(B)=\begin{bmatrix} d^p_{N(B)} & f(B)^{p+1}\\ 0 & -d^{p+1}_{M(B)}\end{bmatrix},\qquad K(f)(b)^p=\begin{bmatrix} N(b)^p & 0\\ 0 & (-1)^{pq}M(b)\end{bmatrix}.$$

for $B,C\in\mathscr{B}$, $p,q\in\mathbf{Z}$, and $b\in\mathscr{B}(B,C)^q$. The category of DG $\mathscr{B}$-modules is denoted by $\mathrm{Dif}\,\mathscr{B}$. It carries an exact structure in the sense of [30] whose admissible short exact sequences are the short exact sequences

$$0\to L\xrightarrow{i}M\xrightarrow{p}N\to 0$$

which split as sequences of graded $\mathscr{B}$-modules.

We denote by $Z^0\mathscr{B}$ the category with the same objects as $\mathscr{B}$ and whose morphisms $A \to B$ correspond bijectively to the elements of

$$Z^0\mathscr{B}(A,B).$$

In the example $\mathscr{B} = \mathscr{C}\mathscr{A}$, the category $Z^0\mathscr{B}$ is the category of chain complexes and morphisms of chain complexes (commuting with the differential).

Clearly, the functor

$$Y : Z^0\mathscr{B} \to \operatorname{Dif}\mathscr{B}, \quad B \mapsto YB = B^\wedge = \mathscr{B}(?,B)$$

is fully faithful (Y stands for 'Yoneda'). A DG $\mathscr{B}$-module is *representable* if it is isomorphic to a functor of the form YB for some $B \in \mathscr{B}$. The category $\mathscr{B}$ is an *exact DG category* if the full subcategory of $\operatorname{Dif}\mathscr{B}$ formed by the representable functors is stable under the translation functors $M \mapsto M[n]$, $n \in \mathbf{Z}$, and closed under extensions. A typical example of an exact DG category is the category $\mathscr{B} = \mathscr{C}\mathscr{A}$.

It is easy to see that each extension of $YA[1]$ by YB in $\operatorname{Dif}\mathscr{B}$ is isomorphic to the mapping cone $\operatorname{Cone}(g)$ of some morphism of DG $\mathscr{B}$-modules $g = Yf : YA \to YB$. Whence the

Lemma. *The category $\mathscr{B}$ is an exact DG category if and only if the following two conditions hold*

(a) *For each $A \in \mathscr{B}$ and each $n \in \mathbf{Z}$, there is an object $A[n]$ in $\mathscr{B}$ and an isomorphism of DG $\mathscr{B}$-modules*

$$Y(A[n]) \xrightarrow{\sim} (YA)[n]$$

(b) *For each morphism $f : A \to B$ of $Z^0\mathscr{B}$, there is an object* $\operatorname{Cone}(f)$ *of $\mathscr{B}$ and an isomorphism of DG $\mathscr{B}$-modules*

$$Y(\operatorname{Cone}(f)) \xrightarrow{\sim} \operatorname{Cone}(Yf).$$

If $\mathscr{B}$ is an exact DG category, then $Z^0\mathscr{B}$ becomes a *Frobenius category* for the exact structure induced from $\operatorname{Dif}\mathscr{B}$, i.e. an exact category with enough injectives, enough projectives and where the classes of projectives and injectives coincide. The *stable category* associated with a Frobenius category is obtained by dividing by the ideal of morphisms factoring through a projective-injective; it is a triangulated category (cf. [12, 11, 21]). By abuse of notation, we will denote the stable category associated with $Z^0\mathscr{B}$ by $\underline{\mathscr{B}}$. In the example of $\mathscr{B} = \mathscr{C}\mathscr{A}$, the stable category $\underline{\mathscr{B}}$ is nothing but the homotopy category of complexes over $\mathscr{A}$.

2.2. *Examples of exact DG categories*

(a) *Categories of complexes.* Let $\mathscr{A}$ be an additive category and $\mathscr{B}$ a full subcategory of the category $\mathscr{C}\mathscr{A}$ of complexes over $\mathscr{A}$ which is closed under (degreewise split) extensions and shifts. Then $\mathscr{B}$ is an exact DG category whose mapping cones are the usual mapping cones of complexes.

(b) *Exact DG subcategories.* Let $\mathscr{B}$ be an exact DG category and $\mathscr{B}'$ an *exact DG subcategory*, i.e. a full DG subcategory such that $Z^0\mathscr{B}'$ is closed in $Z^0\mathscr{B}$ under shifts and extensions. Then $\mathscr{B}'$ is an exact DG category.

(c) *Examples arising from Frobenius categories.* Let $\mathscr{E}$ be a Frobenius category. Let $\mathscr{B}$ be the category of acyclic complexes with projective-injective components over $\mathscr{E}$. This is an exact DG category by example (a). The zero cycle functor induces a triangle equivalence $\underline{\mathscr{B}} \to \underline{\mathscr{E}}$. Hence, up to triangle equivalence, all stable categories of Frobenius categories are obtained as stable categories of exact DG categories.

(d) *Exact envelopes of DG categories.* Let $\mathscr{A}$ be a DG category. Up to equivalence, the category Dif $\mathscr{A}$ contains a unique smallest subcategory containing the YA, $A \in \mathscr{A}$, closed under shifts and (graded split) extensions. This subcategory will be denoted by dgfree $\mathscr{A}$. It is an exact DG category and the functor $\mathscr{A} \to$ dgfree $\mathscr{A}$ is universal among DG functors from $\mathscr{A}$ to exact DG categories. The category dgfree $\mathscr{A}$ may also be constructed more explicitly as follows (cf. also [13, 2.2]): First define $\mathbf{Z}\mathscr{A}$ to be the category whose objects are the pairs (A,r) consisting of an object $A \in \mathscr{A}$ and an integer r; the DG module of morphisms from (A,r) to (B,s) is in bijection with

$$\mathscr{H}om_{\mathscr{A}}(A,B)[s-r].$$

The composition of a morphism $f:(A,r) \to (B,s)$ of degree n with $g:(B,s) \to (C,t)$ of degree m is given by

$$g \circ_{\mathbf{Z}\mathscr{A}} f = (-1)^{n+m+nr+ms} g \circ_{\mathscr{A}} f.$$

Now the objects of dgfree $\mathscr{A}$ are the sequences $(A_1,\ldots,A_n)$ of objects of $\mathbf{Z}\mathscr{A}$ together with matrices $\delta = (\delta_{ij})$ of morphisms $\delta_{ij} \in \mathscr{H}om_{\mathbf{Z}\mathscr{A}}(A_j,A_i)$ such that $\delta_{ij}=0$ for $i \geq j$ and

$$d(\delta_{ij}) + \sum_k \delta_{ik}\delta_{kj} = 0$$

for all i,j. The DG module of morphisms from $(A_1,\ldots,A_n)$ to $(B_1,\ldots,B_m)$ is given by matrices $f=(f_{ij})$, $f_{ij} \in \mathscr{H}om_{\mathbf{Z}\mathscr{A}}(A_j,B_i)$. The differential of a homogeneous morphism f of degree n is defined to be

$$d_{\mathbf{Z}\mathscr{A}} f + \delta \circ f - (-1)^n f \circ \delta,$$

where $d_{\mathbf{Z}\mathscr{A}}$ is applied to each entry of the matrix f and $\delta \circ f$ and $f \circ \delta$ are matrix products. The canonical functor $\Phi : \mathbf{Z}\mathscr{A} \to$ Dif $\mathscr{A}$ sends an object (A,r) to $A^\wedge[r]$ and the canonical functor dgfree $\mathscr{A} \to$ Dif $\mathscr{A}$ sends an object $(A_1,\ldots,A_n)$ to the graded module $\Phi A_1 \oplus \cdots \oplus \Phi A_n$ endowed with the differential $d + \Phi\delta$.

(e) *Functor categories.* Let $\mathscr{A}$ be a small DG category (for example any k-category) and $\mathscr{B}$ an exact DG category. We will define an exact DG category Fun$(\mathscr{A},\mathscr{B})$ whose objects are k-linear DG functors $\mathscr{A} \to \mathscr{B}$. If F and G are two such functors, let $\mathscr{H}om(F,G)^n$ denote the set of homogeneous morphisms of degree n of the underlying graded functors; thus an element φ of $\mathscr{H}om(F,G)^n$ is the datum of a morphism

$\varphi A \in \mathcal{H}om_{\mathcal{B}}(FA, GA)^n$ such that $(Gf)(\varphi A) = (-1)^{nm}(\varphi A')(Ff)$ for each $f \in \mathcal{H}om_{\mathcal{A}}(A, A')^m$, $m \in \mathbf{Z}$. The differential of $\mathcal{H}om(F, G)$ is defined by $(d\varphi)(A) = d(\varphi A)$. Then it is straightforward to check that $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ is an exact DG category (the mapping cone over $\varphi \in Z^0\,\mathcal{H}om(F, G)$ is given by $A \mapsto \mathrm{Cone}(\varphi(A))$).

We define $\mathrm{Rep}(\mathcal{A}, \mathcal{B})$ to be the localization of the stable category of $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ at the class of morphisms $f : F \to G$ such that $fA : FA \to GA$ becomes invertible in $\underline{\mathcal{B}}$ for all $A \in \mathcal{A}$.

(f) *Filtered objects.* Let $F : \mathcal{A} \to \mathcal{B}$ be a DG functor between exact DG categories. Let $\mathrm{Fil}\, F$ be the DG category whose objects are pairs (A, i), where $A \in \mathcal{A}$ and $i : FA \to B$ is an inflation of $Z^0\,\mathcal{B}$. By definition, the DG module of morphisms from $(A, FA \xrightarrow{i} B)$ to $(A', FA' \xrightarrow{i'} B')$ is the pullback of the diagram

$$\begin{array}{ccc} & & \mathcal{H}om_{\mathcal{B}}(B, B') \\ & & \big\downarrow{\scriptstyle i^*} \\ \mathcal{H}om_{\mathcal{A}}(A, A') & \longrightarrow & \mathcal{H}om_{\mathcal{B}}(FA, B'). \end{array}$$

Then $\mathrm{Fil}\, F$ is an exact DG category and a morphism (u, v) of $Z^0\,\mathrm{Fil}\, F$ is invertible in the stable category iff u and v become invertible in the stable categories of $\mathcal{A}$ resp. $\mathcal{B}$.

2.3. *DG functors between exact DG categories*

Let $\mathcal{B}$ and $\mathcal{B}'$ be exact DG categories and let $F : \mathcal{B} \to \mathcal{B}'$ be a DG functor. Recall from [18, 1.1, 1.2] that this means in particular that F is k-linear.

Lemma. *For each $A, B \in \mathcal{B}$, $n \in \mathbf{Z}$ and $f \in Z^0\,\mathcal{B}(A, B)$ there are canonical isomorphisms*

$$F(A[n]) \xrightarrow{\sim} (FA)[n] \quad \textit{and} \quad F(\mathrm{Cone}(f)) \xrightarrow{\sim} \mathrm{Cone}(Ff).$$

Proof. We use the terminology of [18, 6.1]. Define the $\mathcal{B}$-$\mathcal{B}'$-bimodule X by $X(A, B) = \mathcal{B}(A, FB)$ for $A \in \mathcal{B}'$, $B \in \mathcal{B}$. Clearly the functor $T_X : \mathrm{Dif}\,\mathcal{B} \to \mathrm{Dif}\,\mathcal{B}'$ admits canonical isomorphisms as in the claim. Now we have a canonical isomorphism $T_X Y \xrightarrow{\sim} YF$. Whence the claim. $\square$

2.4. *Localization pairs*

A *localization pair* $\mathcal{B}$ is an exact DG category $\mathcal{B}_1$ endowed with a full subcategory $\mathcal{B}_0 \subset \mathcal{B}_1$ such that $Z^0\,\mathcal{B}_0$ is an exact subcategory of $Z^0\,\mathcal{B}_1$ which is stable under shifts and closed under extensions. Then $\underline{\mathcal{B}_0}$ identifies with a full triangulated subcategory of $\underline{\mathcal{B}_1}$. This suffices for the localization $\underline{\mathcal{B}_1}/\underline{\mathcal{B}_0}$ to be well-defined (it is not necessary for $\underline{\mathcal{B}_0}$ to be épaisse; the kernel of the quotient functor $\underline{\mathcal{B}_1} \to \underline{\mathcal{B}_1}/\underline{\mathcal{B}_0}$ is the épaisse closure of $\underline{\mathscr{B}_0}$). By definition, the *triangulated category associated with the localization pair* $\mathscr{B}$ is

$$\mathscr{T}\mathscr{B} := \underline{\mathscr{B}_1}/\underline{\mathscr{B}_0}.$$

For example, if $\mathscr{A}$ is an exact k-linear category, then the pair consisting of $\mathscr{C}^b\mathscr{A}$ and of its subcategory of acyclic complexes is a localization pair and the associated triangulated category is the derived category of $\mathscr{A}$.

Similarly, if A is a DG algebra, then dgfree$A$ endowed with the zero subcategory is a localization pair (which will also be denoted by dgfree$A$). The associated triangulated category is equivalent to a full subcategory of the category per$A$ of perfect objects (= compact objects = small objects [19, 7.10]) in the derived category $\mathscr{D}A$. Moreover, each perfect object is a direct factor (in $\mathscr{D}A$) of an object of dgfree$A$ (this follows from the proof of the theorem of Ravenel–Neeman [29, 31], as explained in [18, 5.2]).

A DG k-module M is *flat* (*resp. closed*) if $M \otimes_k N$ (resp. $\mathscr{H}om_k(M,N)$) is acyclic for each (possibly unbounded) acyclic DG k-module N. A DG category $\mathscr{A}$ is *flat* if $\mathscr{A}(A,B)$ is a flat DG k-module for all $A, B \in \mathscr{A}$. A localization pair $\mathscr{B}$ is *flat* if $\mathscr{B}_0$, and hence $\mathscr{B}_1$, are flat DG categories. Similarly, one defines the notion of *closed DG category* and *closed localization pair*.

The *mixed complex associated with a flat localization pair* $\mathscr{B}$ is the cone

$$C(\mathscr{B}) := \mathrm{Cone}\,(C(\mathscr{B}_0) \to C(\mathscr{B}_1)).$$

The definition of $C(\mathscr{B})$ for an arbitrary localization pair $\mathscr{B}$ will be given in Section 3.9.

If $\mathscr{B}$ and $\mathscr{B}'$ are localization pairs, an *exact functor* $F : \mathscr{B}' \to \mathscr{B}$ is an exact functor $\mathscr{B}_1' \to \mathscr{B}_1$ taking $\mathscr{B}_0'$ to $\mathscr{B}_0$. Such a functor induces a triangulated functor

$$\mathscr{T}\mathscr{B}' \to \mathscr{T}\mathscr{B}$$

and a morphism $C(F) : C(\mathscr{B}') \to C(\mathscr{B})$ in the category of mixed complexes (and therefore in the mixed derived category).

Theorem. (a) *If A is a DG algebra over k, there is a canonical isomorphism* $C(A) \xrightarrow{\sim} C(\mathrm{dgfree}\,A)$ *in the mixed derived category.*

(b) *If $F : \mathscr{B}' \to \mathscr{B}$ is an exact functor between localization pairs such that F induces an equivalence up to factors $\mathscr{T}\mathscr{B}' \to \mathscr{T}\mathscr{B}$, then F induces an isomorphism $C(\mathscr{B}') \xrightarrow{\sim} C(\mathscr{B})$ in the mixed derived category.*

(c) *If we have exact functors between localization pairs*

$$\mathscr{B}' \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{B}''$$

such that the sequence of triangulated categories

$$0 \to \mathscr{T}\mathscr{B}' \to \mathscr{T}\mathscr{B} \to \mathscr{T}\mathscr{B}'' \to 0$$

is exact up to factors, then there is a canonical triangle

$$C(\mathscr{B}') \xrightarrow{C(F)} C(\mathscr{B}) \xrightarrow{C(G)} C(\mathscr{B}'') \xrightarrow{\partial(F,G)} C(\mathscr{B}')[1]$$

in the mixed derived category.

The theorem will be proved in Section 4.11.

2.5. Example: Dual numbers

Let k be a field and $A = k[\varepsilon]/(\varepsilon^2)$. Using the above theorem we will compute $C(\operatorname{mod} A)$. Consider the following categories: the category $\mathscr{B}$ of bounded complexes over $\operatorname{mod} A$; the category $\mathscr{B}'$ of right bounded complexes over $\operatorname{mod} A$ which are acyclic in all degrees $\ll 0$; the smallest full subcategory $\mathscr{B}''$ of $\mathscr{B}'$ closed under shifts and degreewise split extensions and containing the complex

$$P = (\cdots A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \to 0 \to 0 \cdots),$$

where the last component A is in degree 0. Then $\mathscr{B}$ gives rise to a localization pair if we consider it together with its full subcategory of acyclic complexes and similarly for $\mathscr{B}'$ and $\mathscr{B}''$. We denote these localization pairs by the same symbols. Then clearly the inclusion functors

$$\mathscr{B} \to \mathscr{B}' \leftarrow \mathscr{B}''$$

induce equivalences in the associated triangulated categories. Finally, we have $\mathscr{T}\mathscr{B}'' \xleftarrow{\sim} \underline{\mathscr{B}''}$ and this category is generated by P as a triangulated category. Now consider $B = k[T]$ as a differential graded algebra with differential zero where the generator T is of (cohomological) degree 1. Then we have a morphism of DG algebras

$$\varphi : B \to \mathscr{H}om_A(P,P)$$

mapping T to the morphism $P \to P[1]$ which is the identity in all degrees except 0. The morphism φ is in fact a quasi-isomorphism. Hence the functor $Q \mapsto Q \otimes_B P$ induces an equivalence from $\mathscr{T}(\text{degree } B)$ onto $\mathscr{T}\mathscr{B}''$. It follows from the theorem of Section 2.4 that we have isomorphisms

$$C(\operatorname{mod} A) \xleftarrow{\sim} C(\mathscr{B}'') \xleftarrow{\sim} C(B),$$

in the mixed derived category. For example, we have $HH_*(\operatorname{mod} A) \xrightarrow{\sim} HH_*(B) \xrightarrow{\sim} A \otimes B$ as graded k-modules, where T is of (homological) degree -1 and ε of degree 0.

2.6. Example: finite-dimensional algebras

Let k be a field and A a finite-dimensional algebra over k. If A is of finite global dimension, we have $C(\operatorname{mod} A) \xrightarrow{\sim} C(A)$ by the example of Section 1.6. In the general case, let $S_1, \ldots S_n$ be a system of representatives of the simple A-modules and P a projective resolution of the direct sum of the S_i. Put $B = \mathscr{H}om_A(P,P)$. Note that the homology of B is the Ext-algebra

$$\bigoplus_{i,j} \mathrm{Ext}_A^*(S_i, S_j).$$

Then the argument of the example of Section 2.5 shows that we have an isomorphism

$$C(\mathrm{mod}\, A) \xrightarrow{\sim} C(B)$$

in the mixed derived category. J. Rickard asks: Are there finite-dimensional algebras A such that some $HH_n(\mathrm{mod}\, A)$ or $HC_n(\mathrm{mod}\, A)$ is infinite-dimensional ?

2.7. *Mayer–Vietoris squares and triangles*

A triangle functor $F : \mathcal{T} \to \mathcal{T}''$ is a *localization functor* if it induces an equivalence $\mathcal{T}/\ker F \xrightarrow{\sim} \mathcal{T}''$. A square of triangle functors

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{G} & \mathcal{S}'' \\ \downarrow & & \downarrow \\ \mathcal{T} & \xrightarrow{L} & \mathcal{T}'' \end{array}$$

is a *Mayer–Vietoris square* if all four functors are localization functors and the induced triangle functor $\ker G \to \ker L$ is an equivalence.

Theorem. *Let*

$$\begin{array}{ccccc} \mathcal{B}' & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{B}'' \\ {\scriptstyle H'}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H''} \\ \mathcal{C}' & \xrightarrow{K} & \mathcal{C} & \xrightarrow{L} & \mathcal{C}'' \end{array}$$

be a diagram of localization pairs and exact functors such that in the associated triangulated categories, the lines induce exact sequences and the right hand square induces a Mayer–Vietoris square. Then if δ denotes the composition

$$C(\mathcal{C}'') \xrightarrow{\partial(K,L)} C(\mathcal{C}')[1] \xrightarrow{C(H')[1]^{-1}} C(\mathcal{B}')[1] \xrightarrow{C(F)[1]} C(\mathcal{B})[1]$$

(which is well defined in the mixed derived category), the sequence

$$C(\mathcal{B}) \xrightarrow{\alpha} C(\mathcal{B}'') \oplus C(\mathcal{C}) \xrightarrow{\beta} C(\mathcal{C}'') \xrightarrow{\delta} C(\mathcal{B})[1],$$

where

$$\alpha = \begin{bmatrix} -C(G) \\ C(H) \end{bmatrix}, \qquad \beta = [C(H''),\ C(L)],$$

is a triangle (the Mayer–Vietoris triangle *associated with the square).*

The theorem will be proved in Section 4.11. The corresponding Mayer–Vietoris sequence in homology can of course be derived from the homology sequence obtained from the localization theorem. However, the existence of a functorial Mayer–Vietoris *triangle* does not follow from the existence of triangles for localizations.

3. Homotopy and localization

3.1. Homotopy between morphisms of DG algebras

Let k be a commutative ring. Let $\alpha, \beta : A \to B$ be morphisms of DG algebras. An *α-β-derivation of degree r* is a morphism $\Delta : A \to B$ of graded k-modules which is homogeneous of degree r and satisfies

$$\Delta(xy) = \Delta(x)\beta(y) + (-1)^{rn}\alpha(x)\Delta(y)$$

for all $x \in A^n$, $y \in A$. For example, the map $\alpha - \beta$ is an α-β-derivation of degree 0. An *homotopy* from α to β is an α-β-derivation h of degree -1 such that

$$\alpha - \beta = d_B h + h d_A.$$

The morphism α is *homotopic* to β if there is an homotopy from α to β. Clearly homotopy is a bifunctorial equivalence relation on the set of morphisms from A to B. An *homotopy equivalence* is a morphism which becomes invertible after quotienting the category of DG algebras by the homotopy relation.

For a given DG algebra B, there is a 'universal' pair of homotopic morphisms $p_1, p_2 : B' \to B$ constructed as follows: Let $Y = B[-1]$ viewed as a B-B-bimodule. Note that the right action of B on Y is right B-multiplication whereas the left action is twisted left B-multiplication:

$$x.y = (-1)^{pq} xy, \quad y \in B^p, \ x \in B^q$$

and that $d_{B[-1]} = -d_B$. Let B' be the algebra of upper triangular matrices

$$\begin{pmatrix} B & Y \\ 0 & B \end{pmatrix}.$$

View B' as a subalgebra of the graded endomorphism algebra of $B \oplus B[1]$ and as such, endow it with the differential given by the supercommutator with

$$\begin{pmatrix} d & \mathbf{1} \\ 0 & -d \end{pmatrix}.$$

Then B' is a DG algebra (it is a subalgebra of the graded endomorphism algebra of the mapping cone over the identity of B). The two diagonal projections $p_1, p_2 : B' \to B$ are DG algebra morphisms and the map

$$h_0 : \begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix} \mapsto y$$

is an homotopy from p_1 to p_2. It is universal in the sense that if h is an homotopy from α to $\beta : A \to B$, then the map

$$\gamma : a \mapsto \begin{pmatrix} \alpha(a) & h(a) \\ 0 & \beta(a) \end{pmatrix}$$

is a morphism of DG algebras such that $p_1\gamma = \alpha$, $p_2\gamma = \beta$ and $h_0\gamma = h$ and clearly it is the unique morphism with these properties.

Note that both p_i are homotopy equivalences of the underlying chain complexes and that the diagonal map

$$\delta : b \mapsto \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

is a DG algebra morphism satisfying $p_1\,\delta = \mathbf{1}_B = p_2\,\delta$.

Hence if F is a functor defined on the category of DG algebras which inverts morphisms inducing homotopy equivalences of the underlying chain complexes, then $F(\delta)$ is invertible and we have $F(p_1) = F(p_2)$ and hence $F(\alpha) = F(\beta)$ for each pair of homotopic morphisms. In particular, homotopic morphisms $A \to B$ induce the same morphism $C(A) \to C(B)$ in the mixed derived category. We leave it to the reader as an exercise to provide a direct proof of this fact.

3.2. Resolutions of DG algebras

Let k be a commutative ring. Let A be a DG k-algebra. Recall that by definition, A is *flat* if $A \otimes_k N$ is acyclic for each (possibly unbounded) acyclic DG k-module N. This is the case for example if A is *closed* as a DG k-module, i.e. $\mathcal{H}om_k(A, N)$ is acyclic for each acyclic complex N (cf. [19, 7.5]).

A *flat* (*resp. closed*) *resolution* of A is a morphism of DG algebras $\varphi : B \to A$ inducing an isomorphism in homology and such that B is flat (resp. closed) as a DG k-module. Part (a) of the following lemma is well-known for the case of DG algebras concentrated in negative degrees [23, 5.3.6].

Using part (a) of the lemma we define the *mixed complex associated with* A to be $C(A) = C(B)$ where $\varphi : B \to A$ is any flat resolution. Thanks to parts (b) and (c) of the lemma, $C(A)$ is well-defined up to canonical isomorphism in $\mathcal{DM}ix$ and functorial with respect to morphisms of DG algebras. This definition can easily be generalized from DG algebras to small DG categories (Left to the reader. Homotopy is defined only between functors which coincide on objects, and quasi-isomorphisms between small DG k-categories are required to induce bijections between objects).

Note that the lemma shows that in the category of DG algebras and homotopy classes of morphisms, the class of quasi-isomorphisms admits a calculus of right fractions [8] and that the corresponding localization is equivalent to its full subcategory whose objects are the closed DG algebras.

Lemma. (a) *There is a closed* (*hence flat*) *resolution* $\varphi : B \to A$.

(b) *Each diagram of DG algebras*

$$\begin{tikzcd} & B \arrow[d, "\varphi"] \\ A' \arrow[r, "f"] & A \end{tikzcd}$$

where φ is a quasi-isomorphism, can be completed to a square commutative up to homotopy

$$\begin{tikzcd} B' \arrow[r, "f'"] \arrow[d, "\varphi'"'] & B \arrow[d, "\varphi"] \\ A' \arrow[r, "f"] & A \end{tikzcd}$$

where φ' is a quasi-isomorphism.

(c) *If $\alpha, \beta : A \to A'$ and $\gamma : A' \to A''$ are morphisms of DG algebras such that γ is a quasi-isomorphism and $\gamma\alpha$ is homotopic to $\gamma\beta$, then there is a closed resolution $\varphi : B \to A$ such that $\alpha\varphi$ is homotopic to $\beta\varphi$.*

Proof. (a) We endow the category of DG k-modules with the following exact structure (cf. [24, XII; 15]): A sequence

$$0 \to K \to L \to M \to 0$$

is a conflation iff the sequences

$$0 \to K^n \to L^n \to M^n \to 0,$$

and

$$0 \to H^n K \to H^n L \to H^n M \to 0$$

are exact for all $n \in \mathbf{Z}$. Then the class $\mathscr{F}$ formed by all direct sums of shifted copies of the DG modules (=complexes)

$$\cdots 0 \to k \to 0 \cdots \quad \text{and} \quad \cdots 0 \to k \xrightarrow{\mathbf{1}} k \to 0 \cdots$$

contains enough projectives. Choose a deflation $p_0 : V_0 \to A$ with $V_0 \in \mathscr{F}$. Let $B_0 = T(V_0)$ be the DG tensor algebra on V_0 and $\varphi_0 : B_0 \to A$ the morphism extending p_0. Clearly $\varphi_0 : B_0 \to A$ is still a deflation. Choose a morphism $p_1 : V_1 \to B_0$ with $V_1 \in \mathscr{F}$ which induces a deflation onto the kernel of φ_0. Let B_1 be the DG algebra obtained by endowing the free product $B_0 *_k T(V_1[1])$ with the unique differential whose restriction to B_0 is the differential of B_0 and whose restriction to $V_1[1]$ is $-d_{V_1} + p_1$. Let $\varphi_1 : B_1 \to A$ be the unique morphism of algebras whose restriction to B_0 is φ_0 and whose restriction to $V_1[1]$ vanishes. Then φ_1 is compatible with the differential. Continuing in this way we obtain a direct system B_p, $p \in \mathbf{N}$, of DG algebras and a compatible family of morphisms $\varphi_p : B_p \to A$. We let $B = \varinjlim B_p$ and take $\varphi : B \to A$ to be the morphism induced by the φ_p. Then B is closed: indeed, each B_p is closed and B fits into the Milnor triangle

$$\bigoplus_{p\in\mathbf{N}} B_p \xrightarrow{\Phi} \bigoplus_{q\in\mathbf{N}} B_q \xrightarrow{\text{can}} B \to \left(\bigoplus_{p\in\mathbf{N}} B_p\right)[1],$$

where Φ has the components

$$B_p \xrightarrow{[\mathbf{1}\ -\iota]} B_p \oplus B_{p+1} \xrightarrow{\text{can}} \bigoplus_{q\in\mathbf{N}} B_q,\ \iota = \text{incl.}$$

Clearly φ is a deflation. It is easy to see that it induces in fact an isomorphism in homology.

(b) Choose a surjective morphism $p : V \to A$ with contractible $V \in \mathscr{F}$. Then the inclusion $B \to B *_k T(V)$ is an homotopy equivalence and the morphism $B *_k T(V) \to A$ defined by φ and p is a surjective resolution. Therefore we may and will assume that $\varphi : B \to A$ is surjective. Form the pullback diagram

$$\begin{array}{ccc} B' & \xrightarrow{f'} & B \\ {\scriptstyle\varphi'}\downarrow & & \downarrow{\scriptstyle\varphi} \\ A' & \xrightarrow{f} & A. \end{array}$$

Then clearly φ' is a quasi-isomorphism.

(c) We may and will assume that $\gamma : A' \to A''$ is a deflation (cf. the proof of (b). Let $m : A \to A''$ be a homotopy from $\gamma\alpha$ to $\gamma\beta$. As in the proof of (a), we choose a deflation $p_0 : V_0 \to A$ with $V_0 \in \mathscr{F}$, we let $B_0 = T(V_0)$ and take $\varphi_0 : B_0 \to A$ to be the morphism induced by p_0. Since γ is a deflation, the morphism mp_0 equals $\gamma h_0'$ for some $h_0' : V_0 \to A'$ which is homogeneous of degree -1. By construction, the composition of $\alpha p_0 - \beta p_0 - dh_0' - h_0' d$ with γ vanishes. Since $\ker\gamma$ is acyclic and $V_0 \in \mathscr{F}$, there is an $h_0'' : V_0 \to \ker\gamma \subset A'$ such that $\alpha p_0 - \beta p_0 - dh_0' - h_0' d = dh_0'' + h_0'' d$. We put $h_0 = h_0' + h_0''$ and $\alpha_0 = \alpha\varphi_0$, $\beta_0 = \beta\varphi_0$. We extend h_0 to an α_0-β_0-derivation $h_0 : B_0 \to A'$ of degree -1. Now we choose a morphism $p_1 : V_1 \to B_0$ with V_1 in $\mathscr{F}$ inducing a deflation onto $\ker\varphi_0$. We define B_1, φ_1, α_1 and β_1 as in the proof of (a). We will now construct an homotopy h_1 between α_1 and β_1. Note first that we have

$$0 = (\alpha_0 - \beta_0) \circ p_1 = dh_0 p_1 + h_0 dp_1 = d(h_0 p_1) + (h_0 p_1)d.$$

So $h_0 p_1$ defines a morphism of complexes $V_1[1] \to A'$. We claim that its image is contained in $\ker\gamma$. Indeed, we have $\gamma h_0 = m\varphi_0$ since both are $\gamma\alpha_0$-$\gamma\beta_0$-derivations of degree -1 which coincide on V_0. Therefore, we have $\gamma h_0 p_1 = m\varphi_0 p_1 = 0$, as we claimed. Since V_1 belongs to $\mathscr{F}$ and $\ker\gamma$ is acyclic, we can choose a graded morphism $h_1' : V_1[1] \to A'$

of degree -1 such that $dh'_1 + h'_1 d_{V_1[1]} = -h_0 p_1$. We now define $h_1 : B_0 *_k T(V_1[1])$ to be the unique α_1–β_1-derivation of degree -1 which restricts to h_0 on B_0 and to h'_1 on $V_1[1]$. It is then easy to check that $\alpha_1 - \beta_1$ and $dh_1 + h_1 d$ coincide on B_0 and $V_1[1]$ and hence on $B_1 = B_0 *_k T(V_1[1])$.

Continuing in this way, we obtain a direct system as in the proof of a), and in addition we have a compatible family of graded morphisms h_p of degree -1. We define $B = \varinjlim B_p$ and let $\varphi : B \to A$ be the morphism induced by the φ_p. Then the morphism induced by the h_p yields a homotopy between $\alpha\varphi$ and $\beta\varphi$. □

3.3. Homotopy between functors

Let k be a commutative ring and $\mathscr{A}$, $\mathscr{B}$ small exact DG categories. Let $F, G : \mathscr{A} \to \mathscr{B}$ be DG functors.

An *a-homotopy* from F to G is the datum of a morphism of DG functors $\alpha : F \to G$ such that αA is an inflation of $Z^0\mathscr{B}$ which becomes invertible in $H^0\mathscr{B}$ for all $A \in \mathscr{A}$.

A *b-homotopy* from F to G is the datum of

- a morphism $\eta A : FA \to GA$ of $Z^0\mathscr{B}$ which becomes invertible in $H^0\mathscr{B}$ for all $A \in \mathscr{A}$ (but which will not be functorial in A, in general)
- a morphism of graded k-modules homogeneous of degree -1

$$h = h(A, B) : \mathscr{A}(A, B) \to \mathscr{B}(FA, GB)$$

for all $A, B \in \mathscr{A}$ such that we have

$$(\eta B)(Ff) - (Gf)(\eta A) = d(h(f)) + h(d(f)),$$
$$h(fg) = h(f)(Fg) + (-1)^n (Gf) h(g)$$

for all composable morphisms f, g of $\mathscr{A}$, where f is of degree n.

A *c-homotopy* from F to G is an isomorphism from F to G in $\mathrm{Rep}(\mathscr{A}, \mathscr{B})$.

Let $x = a, b$ or c. We write $\sim_x$ for the smallest equivalence relation containing all pairs (F, G) such that there is an x-homotopy from F to G. Using statement (a) of the following lemma, we define F to be *homotopic* to G if we have $F \sim_x G$ for $x = a, b, c$.

Lemma. (a) *The relations $\sim_a$, $\sim_b$ and $\sim_c$ coincide.*

(b) *There is a universal pair $P_1, P_2 : \mathscr{B}' \to \mathscr{B}$ of a-homotopic DG functors (i.e. for each pair $F, G : \mathscr{A} \to \mathscr{B}$ of a-homotopic functors there is a functor $\Phi : \mathscr{A} \to \mathscr{B}$ such that $P_1\Phi = F$, $P_2\Phi = G$). Moreover, there is a DG functor $D : \mathscr{B} \to \mathscr{B}'$ such that $P_1 D = \mathbf{1}_{\mathscr{B}} = P_2 D$ and D induces an equivalence in the associated stable categories and an isomorphism of DG k-modules*

$$\mathscr{B}(A, B) \to \mathscr{B}'(DA, DB)$$

for all $A, B \in \mathscr{B}$.

Proof. (a) If $\alpha : F \to G$ is an a-homotopy, then by putting $\eta A = \alpha A$ and $h(A,B) = 0$ for all $A, B \in \mathscr{A}$ we obtain a b-homotopy. Now suppose that we have an arbitrary b-homotopy (η, h) from F to G. Consider the sequences

$$GA \to IA \to (FA)[1], \quad A \in \mathscr{A}$$

where $IA = \mathrm{Cone}(\eta A)$. By the assumption on ηA, the term IA is a zero object of $\underline{\mathscr{B}}$. We will now make $A \mapsto IA$ into a DG functor $\mathscr{A} \to \mathscr{B}$ such that the above sequence becomes a sequence of DG functors. This will clearly imply that F is isomorphic to G in $\mathrm{Rep}(\mathscr{A}, \mathscr{B})$. To define the morphisms

$$\mathscr{A}(A,B) \to \mathscr{B}(IA, IB)$$

we identify $\mathscr{B}(IA, IB)$ with the module of matrices

$$\begin{bmatrix} \mathscr{B}(GA,GB) & \mathscr{B}(FA[1],GB) \\ \mathscr{B}(GA,FB[1]) & \mathscr{B}(FA[1],FB[1]) \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} \mathscr{B}(GA,GB) & \mathscr{B}(FA,GB)[-1] \\ \mathscr{B}(GA,FB[1]) & \mathscr{B}(FA,FB) \end{bmatrix}.$$

Then the morphism $\mathscr{A}(A,B) \to \mathscr{B}(IA,IB)$ is given by

$$f \mapsto \begin{bmatrix} Gf & h(f) \\ 0 & Ff \end{bmatrix}.$$

It is easy to check that this defines a morphism of complexes and a functor which is moreover compatible with the above sequences.

Finally, suppose that F is isomorphic to G in $\mathrm{Rep}(\mathscr{A}, \mathscr{B})$. Then we apply Lemma 3.5 below to $\mathscr{B} = \mathrm{Fun}(\mathscr{A}, \mathscr{B})$ and to the class Σ of morphisms φ such that φA is invertible in $\underline{\mathscr{B}}$ for all $A \in \mathscr{A}$. We conclude that there is a finite sequence connecting F to G and consisting of inflations of $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ which become invertible in $\mathrm{Rep}(\mathscr{A}, \mathscr{B})$. This clearly implies $F \sim_a G$.

(b) Let $\mathscr{B}'$ be the full exact DG subcategory of $\mathrm{Fil}\,\mathscr{B}$ (cf. Example 2.2(f)) whose objects are the inflations $i : B_1 \to B_2$ which become invertible in $\underline{\mathscr{B}}$. The DG functors

$$P_j : \mathscr{B}' \to \mathscr{B}, \qquad i \mapsto B_j, \; j = 1, 2,$$

$$D : \mathscr{B} \to \mathscr{B}', \qquad B \mapsto (B \xrightarrow{\mathbf{1}} B).$$

satisfy the claim. Let α be an a-homotopy from F to G. Then clearly

$$\Phi : \mathscr{A} \to \mathscr{B}', \quad A \mapsto (FA \xrightarrow{\alpha A} GA)$$

satisfies $P_1 \Phi = F$, $P_2 \Phi = G$. $\square$

3.4. Homotopy invariance (exact DG categories)

Let $\mathscr{A}$ and $\mathscr{B}$ be small exact DG categories and $F : \mathscr{A} \to \mathscr{B}$ a DG functor. We say that F is a *resolution* if F induces an equivalence $\underline{\mathscr{A}} \to \underline{\mathscr{B}}$. It is a *pure* resolution if moreover F induces a quasi-isomorphism

$$\mathscr{A}(A,B)\otimes V\to\mathscr{B}(FA,FB)\otimes V$$

for each DG k-module V, and all $A,B\in\mathscr{B}$. Note that if F is a resolution and $\mathscr{A}$, $\mathscr{B}$ are flat, then F is automatically a pure resolution.

Lemma. (a) *If F is a pure resolution, then F induces an isomorphism $C(\mathscr{A})\to C(\mathscr{B})$ in $\mathscr{D}\mathscr{M}ix$.*

(b) *Two homotopic DG functors $\mathscr{A}\to\mathscr{B}$ induce the same morphism $C(\mathscr{A})\to C(\mathscr{B})$ in $\mathscr{D}\mathscr{M}ix$.*

Proof. (a) Let $\mathscr{A}'$ be the full DG category of $\mathscr{B}$ formed by the objects FA, $A\in\mathscr{A}$. The assumption that F induces a quasi-isomorphism

$$\mathscr{A}(A,B)\otimes V\to\mathscr{B}(FA,FB)\otimes V$$

for each DG k-module V, and all $A,B\in\mathscr{B}$ implies that F induces a quasi-isomorphism $C(\mathscr{A})\to C(\mathscr{A}')$. The DG category version of Lemma 1.2 of [19] implies that the inclusion $\mathscr{A}'\subset\mathscr{B}$ induces a quasi-isomorphism $C(\mathscr{A}')\to C(\mathscr{B})$.

(b) This follows from (a) and Lemma 3.3(b). □

3.5. *Isomorphisms in localizations*

If $\mathscr{C}$ is a small category, we denote by Iso $\mathscr{C}$ the category with the same objects as $\mathscr{C}$ and whose morphisms are the isomorphisms of $\mathscr{C}$ and by Quot $\mathscr{C}$ the localization of $\mathscr{C}$ at the class of all morphisms.

Let $\mathscr{E}$ be a Frobenius category and Σ a multiplicative system [34] in $\underline{\mathscr{E}}$. Let $\mathscr{I}_\Sigma$ denote the category whose objects are those of $\mathscr{E}$ and whose morphisms are the inflations of $\mathscr{E}$ which become invertible in $\underline{\mathscr{E}}[\Sigma^{-1}]$.

Lemma. *The canonical functor* $\mathrm{Quot}(\mathscr{I}_\Sigma)\to\mathrm{Iso}(\underline{\mathscr{E}}[\Sigma^{-1}])$ *is an isomorphism of categories.*

Proof. Let $\underline{\mathscr{E}}_\Sigma$ be the category whose objects are those of $\mathscr{E}$ and whose morphisms are those of $\underline{\mathscr{E}}$ which become invertible in $\underline{\mathscr{E}}[\Sigma^{-1}]$. Then, since Σ admits a calculus of fractions, the canonical functor $\mathrm{Quot}(\underline{\mathscr{E}}_\Sigma)\to\mathrm{Iso}(\underline{\mathscr{E}}[\Sigma^{-1}])$ is an isomorphism. So it remains to be proved that the canonical functor $\mathrm{Quot}(\mathscr{I}_\Sigma)\to\mathrm{Quot}(\underline{\mathscr{E}}_\Sigma)$ is an isomorphism. Let Ω denote the class of morphisms of $\mathscr{I}_\Sigma$ which become invertible in $\underline{\mathscr{E}}$. Clearly $\mathrm{Quot}(\mathscr{I}_\Sigma)$ identifies with $\mathrm{Quot}(\mathscr{I}_\Sigma[\Omega^{-1}])$. So it is enough to prove that the canonical functor $F:\mathscr{I}_\Sigma[\Omega^{-1}]\to\underline{\mathscr{E}}_\Sigma$ is an isomorphism. We will construct an inverse G to this functor.

Let $Q:\mathcal{I}_\Sigma \to \mathcal{I}_\Sigma[\Omega^{-1}]$ be the quotient functor. Let $\overline{f}:X\to Y$ be a morphism of $\underline{\mathcal{E}}_\Sigma$. Choose an inflation $i:X\to I$ with injective I. Consider the morphisms

$$\begin{bmatrix} f \\ i \end{bmatrix}:X\to Y\oplus I \quad \text{and} \quad \begin{bmatrix} \mathbf{1}_Y \\ 0 \end{bmatrix}:X\to Y\oplus I.$$

Define

$$G(\overline{f})=Q\left(\begin{bmatrix} \mathbf{1}_Y \\ 0 \end{bmatrix}\right)^{-1}\circ Q\left(\begin{bmatrix} f \\ i \end{bmatrix}\right).$$

Let us show that this does not depend on the choice of the inflation i. Indeed, let $i':X\to I'$ be another inflation with injective I'. The claim will follow once we prove that the following diagram becomes commutative after applying Q

$$\begin{tikzcd}
 & Y\oplus I \arrow[d, "{[\iota_1\ \iota_2]}"] & \\
X \arrow[ur, "{[f\ i]^t}"] \arrow[r, "{[f\ i\ i']^t}"] \arrow[dr, "{[f\ i']^t}"'] & Y\oplus I\oplus I' & Y \arrow[ul, "\iota_Y"'] \arrow[l, "\iota_Y"'] \arrow[dl, "\iota_Y"] \\
 & Y\oplus I' \arrow[u, "{[\iota_1\ \iota_3]}"'] &
\end{tikzcd}$$

We have to prove that

$$Q\left(\begin{bmatrix} f \\ i \\ 0 \end{bmatrix}\right)=Q\left(\begin{bmatrix} f \\ i \\ i' \end{bmatrix}\right)=Q\left(\begin{bmatrix} f \\ 0 \\ i' \end{bmatrix}\right).$$

To show the first equality, choose $j:I\to I'$ such that $i'=ji$. Then $[f\ \ i\ \ i']^t$ is the composition of $[f\ i\ 0]^t$ followed by

$$s=\mathbf{1}_Y\oplus\begin{bmatrix} \mathbf{1}_I & 0 \\ j & \mathbf{1}_{I'} \end{bmatrix}.$$

The functor Q maps s to the identity of $Y\oplus I\oplus I'$ since its composition with ι_Y is the identity and $Q(\iota_Y)$ is invertible. The second equality is proved similarly. Now let us show that $G(\overline{g}\overline{f})=G(\overline{g})G(\overline{f})$. Indeed, this now follows from the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "{[f\ i]^t}"] & Y \oplus I \arrow[r, "u"] & Z \oplus I \oplus J \\ & Y \arrow[u, "\iota_Y"] \arrow[r, "{[g\ j]^t}"] & Z \oplus J \arrow[u, "v"'] \\ & & Z \arrow[u, "\iota_Z"'] \end{tikzcd}$$

where $j : Y \to J$ is an inflation with injective J and

$$u = \begin{bmatrix} g & 0 \\ 0 & \mathbf{1}_I \\ j & 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} \mathbf{1}_Z & 0 \\ 0 & 0 \\ 0 & \mathbf{1}_J \end{bmatrix}.$$

The fact that $G(\mathbf{1}_X) = \mathbf{1}_{GX}$ follows from $G(\mathbf{1}_X) = G(\mathbf{1}_X^2) = G(\mathbf{1}_X)^2$ because $G(\mathbf{1}_X)$ is invertible (it is a composition of two isomorphisms of $\underline{\mathscr{E}}$.). This means that we have

$$Q\left(\begin{bmatrix} \mathbf{1}_X \\ i \end{bmatrix}\right) = Q\left(\begin{bmatrix} \mathbf{1}_X \\ 0 \end{bmatrix}\right)$$

if $f : X \to Y$ and $i : X \to I$ is an inflation with injective I. We will use this to show that $G(\overline{f}) = Q(f)$ if f is an inflation which becomes invertible in $\underline{\mathscr{E}}[\Sigma^{-1}]$. Indeed, in this case, we have $i = jf$ for some $j : Y \to I$. Therefore

$$\begin{bmatrix} f \\ i \end{bmatrix} = \begin{bmatrix} f \\ jf \end{bmatrix} = \begin{bmatrix} \mathbf{1}_Y \\ j \end{bmatrix} \circ f$$

and

$$Q\left(\begin{bmatrix} f \\ i \end{bmatrix}\right) = Q\left(\begin{bmatrix} \mathbf{1}_Y \\ j \end{bmatrix}\right) \circ Q(f) = Q\left(\begin{bmatrix} \mathbf{1}_Y \\ 0 \end{bmatrix}\right) \circ Q(f)$$

which implies that

$$G(\overline{f}) = Q\left(\begin{bmatrix} \mathbf{1}_Y \\ 0 \end{bmatrix}\right)^{-1} \circ Q\left(\begin{bmatrix} f \\ i \end{bmatrix}\right) = Q(f).$$

If follows that G is full. But clearly, the composition FG is the identity of $\underline{\mathscr{E}}_\Sigma$. Thus, F and G are inverse isomorphisms. □

3.6. *Resolutions of exact DG categories*

Let k be a commutative ring and $\mathscr{A}$ a small exact DG category. A DG functor $F : \mathscr{B} \to \mathscr{A}$ is a *flat* (*resp. closed*) *resolution* if it is a resolution Section 3.4 and $\mathscr{B}$ is a flat (resp. closed) exact DG category (Section 2.4).

By definition, the *mixed complex* associated with $\mathscr{A}$ is $C(\mathscr{B})$, where $\mathscr{B} \to \mathscr{A}$ is a flat resolution. By Lemma 3.4 and the following lemma, this is well-defined up to canonical isomorphism in $\mathscr{DM}ix$ and functorial with respect to $\mathscr{A}$.

Lemma. (a) *Each exact DG category $\mathscr{A}$ admits a closed resolution $\mathscr{B} \to \mathscr{A}$.*

(b) *Each diagram of exact DG categories and DG functors*

$$\begin{tikzcd} & \mathscr{B} \arrow[d, "F"] \\ \mathscr{A}' \arrow[r, "G"] & \mathscr{A} \end{tikzcd}$$

where F is a resolution, may be completed to a square

$$\begin{tikzcd} \mathscr{B}' \arrow[r, "G'"] \arrow[d, "F'"'] & \mathscr{B} \arrow[d, "F"] \\ \mathscr{A}' \arrow[r, "G"] & \mathscr{A} \end{tikzcd}$$

which is commutative up to homotopy and where F' is a resolution.

(c) *Suppose that $K, L : \mathscr{A} \to \mathscr{A}'$ are DG functors and there is a DG functor $G : \mathscr{A}' \to \mathscr{A}''$ inducing a stable equivalence $\underline{\mathscr{A}'} \xrightarrow{\sim} \underline{\mathscr{A}''}$ and such that GK is homotopic to GL. Then there is a closed resolution $F : \mathscr{B} \to \mathscr{A}$ such that KF is homotopic to LF.*

Proof. (a) We imitate the proof of Section 3.2(a): We choose a deflation

$$p_0 = p_0(A,B) : V_0(A,B) \to \mathscr{A}(A,B)$$

with $V_0(A,B) \in \mathscr{F}$ for all $A, B \in \mathscr{A}$. We let $\mathscr{B}_0 = T(V_0)$ be the category with the same objects as $\mathscr{A}$ and whose morphisms $A \to B$ are parametrized by the direct sum of the

$$V_0(B_{n-1},B) \otimes V_0(B_{n-2},B_{n-1}) \otimes \cdots \otimes V_0(A,B_1),$$

where $B_1, \ldots, B_{n-1}$ runs through all finite sequences of objects of $\mathscr{A}$, $n \geq 0$. We let $F_0 : \mathscr{B}_0 \to \mathscr{A}$ be the functor extending p_0 etc.

This construction yields a DG functor $F' : \mathscr{B}' \to \mathscr{A}$ which is bijective on objects and induces quasi-isomorphisms

$$\mathscr{B}'(A,B) \to \mathscr{A}(F'A, F'B)$$

for all $A, B \in \mathscr{B}'$. The DG category $\mathscr{B}'$ is closed but will not be exact, in general. We put $\mathscr{B} = \operatorname{dgfree} \mathscr{B}'$ and let $F : \mathscr{B} \to \mathscr{A}$ be the functor induced by F' (cf. Example 2.2(c)).

(b) In a first step, we will replace F by a functor which induces deflations in the morphism spaces and a surjection of the object sets. Indeed, let $\mathscr{B}'$ be the full subcategory of $\mathrm{Fil}\,F$ (cf. Example 2.2(e)) formed by the pairs $(B,\ i: FB \to A)$ where i becomes invertible in $\underline{\mathscr{B}}$. Then the functors

$$\mathscr{B} \to \mathscr{B}',\quad B \longmapsto (B, \mathbf{1}: FB \to FB)$$
$$\mathscr{B}' \to \mathscr{B},\quad (B, i: FB \to A) \mapsto B$$

are inverse to each other up to homotopy and the functor

$$F': \mathscr{B}' \to \mathscr{A},\quad (B, i: FB \to A) \mapsto A$$

is surjective on objects. Using the method of the proof of Section 3.2(b) we can modify it so as to induce deflations of the morphism spaces. Let us therefore assume that F is surjective on objects and induces deflations of the morphism spaces. For each object A' of $\mathscr{A}'$, we choose a preimage $G'A'$ of GA' under F. Now let $\mathscr{B}'$ be the category with the same objects as $\mathscr{A}'$ and whose morphism spaces are given by pullback diagrams

$$\begin{tikzcd}
\mathscr{B}'(A_1', A_2') \arrow[r] \arrow[d] & \mathscr{B}'(G'A_1', G'A_2') \arrow[d] \\
\mathscr{A}'(A_1', A_2') \arrow[r] & \mathscr{A}(GA_1', GA_2').
\end{tikzcd}$$

Then $\mathscr{B}'$ is an exact DG category and the obvious functors $F': \mathscr{B}' \to \mathscr{A}'$ and $G': \mathscr{B}' \to \mathscr{B}$ yield a commutative diagram as in the claim.

(c) As in the proof of (b) we may and will assume that K induces surjections of the morphism sets and deflations in the morphism spaces. Using Section 3.3 we may assume that we have a b-homotopy $(\eta, h): GK \to GL$.

For all $A, B \in \mathscr{A}$, choose a deflation $p_0: V_0(A,B) \to \mathscr{A}(A,B)$ and let $\mathscr{B}_0 = T(V_0)$ and $F_0: \mathscr{B}_0 \to \mathscr{A}$ be the functor extending p_0 as in the proof of (a).

$$\mathscr{B}_0 = T(V_0) \xrightarrow{F_0} \mathscr{A} \xrightarrow{K,L} \mathscr{A}' \xrightarrow{G} \mathscr{A}''$$

For each $A \in \mathscr{B}_0$, choose a morphism

$$\tilde{\eta}A : KF_0A \to LF_0A$$

of $Z^0\mathscr{A}$ such that $G(\tilde{\eta}A) = \eta A$ (this is possible, since G induces deflations in the morphism spaces). Let $A, B \in \mathscr{B}_0$. Since G induces a deflation, there is a morphism h_0' such that the following diagram becomes commutative

$$\begin{tikzcd}
V_0(A,B) \arrow[r, "h_0'"] \arrow[d, "p_0"'] & \mathscr{A}'(KA, LB) \arrow[d] \\
\mathscr{A}(A,B) \arrow[r, "h"] & \mathscr{A}''(GKA, GLB).
\end{tikzcd}$$

Then the morphism

$$V_0(A,B) \to \mathscr{A}'(KA,LB), \qquad a \mapsto (\tilde{\eta}B)(Ka) - (La)(\tilde{\eta}A) - d(h_0'(a)) - h_0'(d(a))$$

factors through the acyclic subcomplex $\ker G \subset \mathscr{A}'(KA,LB)$. Since $V_0(A,B)$ belongs to $\mathscr{F}$, it equals $dh_0'' + h_0''d$ for some $h_0'' : V_0(A,B) \to \ker G \subset \mathscr{A}'(KA,LB)$. We put $h_0 = h_0' + h_0''$. We can then extend h_0 uniquely to $\mathscr{B}_0$ in such a way that $(\tilde{\eta}, h_0)$ defines a b-homotopy from KF_0 to LF_0. Now we construct $\mathscr{B}_1$ as in the proof of (a) by choosing morphisms $p_1 : V_1(A,B) \to \mathscr{B}_0(A,B)$ inducing deflations onto the kernel of

$$\mathscr{B}_0(A,B) \to \mathscr{A}(A,B)$$

and letting $\mathscr{B}_1 = \mathscr{B}_0 *_k T(V_1[1])$. To extend h_0 to $\mathscr{B}_1$ we first note that we have

$$0 = (\tilde{\eta}B)(KF_0 f) - (LF_0 f)(\tilde{\eta}A) = d(h_0(f)) + h_0(d(f))$$

for f belonging to the kernel of F_0. So $h_0 p_1$ defines a morphism of complexes $V_1[1] \to \mathscr{A}'(KA,LB)$. Moreover, its composition with G vanishes.

$$\begin{tikzcd}
V_1(A,B) \arrow[r] \arrow[d, "p_1"'] & \ker G \arrow[d] \\
\mathscr{B}_0(A,B) \arrow[r, "h_0"] \arrow[d, "F_0"'] & \mathscr{A}'(KA,LB) \arrow[d, "G"] \\
\mathscr{A}(A,B) \arrow[r, "h"'] & \mathscr{A}''(GKA,GLB)
\end{tikzcd}$$

So it factors through $\ker G$, which is acyclic, and since $V_1[1]$ belongs to $\mathscr{F}$, we have

$$h_0 p_1 = dh_1' + h_1' d_{V_1[1]}$$

for some morphism $h_1' : V_1[1] \to \mathscr{A}'(KA,LB)$ of degree -1. We can then define a unique b-homotopy (η, h_1) from KF_1 to $LF_1 : \mathscr{B}_1 \to \mathscr{A}'$ by requiring that h_1 restricts to h_0 on $\mathscr{B}_0$ and to h_1' on $V_1[1]$. By iterating this construction we obtain a direct system of DG categories $\mathscr{B}_p$ and a compatible family of b-homotopies (η, h_p) from KF_p to LF_p. Passage to the limit yields the result. $\square$

3.7. Morphisms of mixed complexes

Let $\mathscr{M}or\mathscr{M}ix$ be the category of morphisms $C_1 \to C_2$ of mixed complexes. If we identify a mixed complex with a DG Λ-module as in Section 1.2, a morphism $f : C_1 \to C_2$ of mixed complexes may be identified with the DG module $C_1 \oplus C_2$ over the DG algebra

$$\begin{pmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{pmatrix},$$

where e_{12} acts by $(c_1, c_2) \mapsto (0, f(c_1))$. Using this identification we view $\mathcal{M}or\mathcal{M}ix$ as a category of DG modules and we define $\mathcal{D}\mathcal{M}or\mathcal{M}ix$ to be the associated derived category. Note that a morphism of $\mathcal{D}\mathcal{M}or\mathcal{M}ix$ is invertible iff both of its components are invertible in $\mathcal{D}\mathcal{M}ix$.

3.8. Homotopy invariance (localization pairs)

Let k be a commutative ring. Let $F, G : \mathcal{B} \to \mathcal{B}'$ be exact functors between localization pairs. By definition, F is *homotopic* to G if the underlying exact functors $\mathcal{B}_1 \to \mathcal{B}'_1$ are homotopic (Section 3.3). The functor F is a *pure* (*resp. closed, resp. flat*) *resolution* if this holds for the induced functors $\mathcal{B}_1 \to \mathcal{B}'_1$ and $\mathcal{B}_0 \to \mathcal{B}'_0$.

For a localization pair $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$, the object $Cm(\mathcal{B}) \in \mathcal{D}\mathcal{M}or\mathcal{M}ix$ is defined to be the morphism

$$C(\mathcal{B}_0) \to C(\mathcal{B}_1)$$

of $\mathcal{M}ix$. This is clearly functorial in $\mathcal{B}$.

Lemma. (a) *If F is a pure resolution, then F induces an isomorphism $Cm(\mathcal{B}) \to Cm(\mathcal{B}')$ in $\mathcal{D}\mathcal{M}or\mathcal{M}ix$.*

(b) *If $F, G : \mathcal{B} \to \mathcal{B}'$ are homotopic, they induce the same morphism $Cm(\mathcal{B}) \to Cm(\mathcal{B}')$ in $\mathcal{D}\mathcal{M}or\mathcal{M}ix$.*

Proof. Statement (a) follows from Section 3.4 and the fact that a morphism of $\mathcal{D}\mathcal{M}or\mathcal{M}ix$ is invertible iff its two components are invertible.

(b) Suppose that we have an a-homotopy between $F : \mathcal{B}_1 \to \mathcal{B}'_1$ and $F' : \mathcal{B}_1 \to \mathcal{B}'_1$. Then we do not necessarily have $F'(\mathcal{B}_0) \subset \mathcal{B}'_0$. However, suppose that $\mathcal{B}'_0$ is *saturated* in $\mathcal{B}'_1$, i.e. the image of $\mathcal{B}'_0$ in $\underline{\mathcal{B}'_1}$ is closed under isomorphisms. Then we do have $F'(\mathcal{B}_0) \subset \mathcal{B}'_0$. Now for any $\mathcal{B}'_0 \subset \mathcal{B}'_1$, there is a saturation $sat(\mathcal{B}'_0) \subset \mathcal{B}'_1$ and by (a), the inclusion yields an isomorphism $Cm(\mathcal{B}'_0, \mathcal{B}'_1) \xrightarrow{\sim} Cm(sat(\mathcal{B}'_0), \mathcal{B}'_1)$. So we may assume that $\mathcal{B}'_0$ is saturated and then the claim is proved by a variant of the proof of Lemma 3.4(b). □

3.9. Resolutions of localization pairs

If $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ is a localization pair, a (*flat resp. closed*) *resolution* of $\mathcal{A}$ is a morphism of localization pairs $\mathcal{B} \to \mathcal{A}$ such that $\mathcal{B}_1 \to \mathcal{A}_1$ and $\mathcal{B}_0 \to \mathcal{A}_0$ are (flat resp. closed) resolutions of exact DG categories.

Using part (a) of the following lemma, for a localization pair $\mathcal{A}$, we define the object $Cm(\mathcal{A})$ to be $Cm(\mathcal{B})$, where $\mathcal{B} \to \mathcal{A}$ is a flat resolution. Thanks to Section 3.8 and the lemma, this is well-defined up to canonical isomorphism in $\mathcal{D}\mathcal{M}or\mathcal{M}ix$ and functorial in $\mathcal{A}$. The *mixed complex* associated with $\mathcal{A}$ is defined to be $\mathrm{Cone}(Cm(\mathcal{A}))$.

If $\mathcal{A}$ is a (not necessarily flat) exact category, the mixed complex of $\mathcal{A}$ is defined to be the mixed complex associated with the localization pair $\mathcal{A}c^b\mathcal{A} \subset \mathcal{C}^b\mathcal{A}$.

Lemma. (a) *For each localization pair $\mathcal{A}$, there is a closed resolution $F : \mathcal{B} \to \mathcal{A}$.*

(b) *Each diagram of localization pairs*

$$\begin{tikzcd} & \mathscr{B} \arrow[d, "F"] \\ \mathscr{A}' \arrow[r, "G"] & \mathscr{A} \end{tikzcd}$$

where F is a resolution, may be completed to a square

$$\begin{tikzcd} \mathscr{B}' \arrow[r, "G'"] \arrow[d, "F'"'] & \mathscr{B} \arrow[d, "F"] \\ \mathscr{A}' \arrow[r, "G"] & \mathscr{A} \end{tikzcd}$$

where F' is a resolution and GF' is homotopic to FG'.

(c) *Suppose that $K, L : \mathscr{A} \to \mathscr{A}'$ are exact functors between localization pairs and there is an exact functor $G : \mathscr{A}' \to \mathscr{A}''$ which is a resolution of $\mathscr{A}''$ and such that GK is homotopic to GL. Then there is a resolution $F : \mathscr{B} \to \mathscr{A}$ such that KF is homotopic to LF.*

Proof. This is a straightforward consequence of Lemmas 3.6 and 3.8. For example, to prove (a), we choose a flat resolution $F_1 : \mathscr{B}_1 \to \mathscr{A}_1$ and let $\mathscr{B}_0$ be the preimage of $\mathscr{A}_0$ under F_1. □

4. Completion, cokernels

4.1. The categories $\mathscr{M}^b$ and $\mathscr{M}$

Let **U** be a universe containing an infinite set. A category $\mathscr{C}$ is a **U**-*category* if it is small and the set $\mathscr{C}(X, Y)$ belongs to **U** for all $X, Y \in \mathscr{C}$. It is **U**-*small* if the set of its morphisms belongs to **U**. It *has* **U**-*coproducts* if each family $(X_i)_{i \in I}$ of objects of $\mathscr{C}$ indexed by a set I of **U** admits a coproduct in $\mathscr{C}$.

Fix k a commutative ring belonging to **U**. The '*strict*' *category* $\mathscr{M}^b_{str}$ has as objects the **U**-small exact DG categories. Its morphisms are the DG functors. The *homotopy category* $\mathscr{M}^b_{htp}$ is obtained from $\mathscr{M}^b_{str}$ by identifying homotopic DG functors. Finally, the category $\mathscr{M}^b$ is obtained from $\mathscr{M}^b_{htp}$ by localization at the class of DG functors inducing equivalences in the stable categories. In Section 3.6, we have shown that this class admits a calculus of right fractions [8] and we have constructed the cyclic functor

$$C : \mathscr{M}^b \to \mathscr{D}\mathscr{M}ix, \quad \mathscr{A} \mapsto C(\mathscr{A}).$$

A triangulated category $\mathscr{T}$ is *compactly* $\mathbf{U}$*-generated* if it is a $\mathbf{U}$-category having $\mathbf{U}$-coproducts and admitting a family $(X_i)_{i\in I}$, $I\in\mathbf{U}$, of compact objects such that $\mathscr{T}$ coincides with its smallest triangulated subcategory containing all X_i and stable under $\mathbf{U}$-coproducts. In this case, the family (X_i) is a *family of* $\mathbf{U}$*-generators* of $\mathscr{T}$. Such a category is *Karoubian* (as shown in [2]), i.e. idempotents split in $\mathscr{T}$.

Let $\mathscr{A}$ be a $\mathbf{U}$-small DG category. Let $\mathscr{A}^+$ be the DG category of all DG modules P such that P is closed and $P(A)$ belongs to $\mathbf{U}$ for all $A\in\mathscr{A}$. Then $\mathscr{A}^+$ is an exact DG category and the associated stable category $\underline{\mathscr{A}^+}$ is a compactly $\mathbf{U}$-generated triangulated category. This results by inspection of the proofs from Section 3 of [18].

The assignment $\mathscr{A}\mapsto\mathscr{A}^+$ is functorial in the following sense: Let $\mathscr{A},\mathscr{B}$ be $\mathbf{U}$-small DG categories and $F:\mathscr{A}\to\mathscr{B}$ a DG functor. Then the tensor product by the bimodule $(A,B)\mapsto\mathscr{B}(B,FA)$ yields an induced functor $F^+:\mathscr{A}^+\to\mathscr{B}^+$ well-defined up to canonical isomorphism. The associated functor $\underline{\mathscr{A}^+}\to\underline{\mathscr{B}^+}$ preserves compactness and commutes with arbitrary coproducts. If F induces an equivalence in the stable categories, then so does F^+.

We denote by $\mathscr{M}_{str}$ the category whose objects are the exact DG $\mathbf{U}$-categories $\mathscr{A}$ such that $\underline{\mathscr{A}}$ is compactly $\mathbf{U}$-generated. The morphisms of $\mathscr{M}_{str}$ are DG functors $F:\mathscr{A}\to\mathscr{B}$ such that the induced functor $\underline{\mathscr{A}}\to\underline{\mathscr{B}}$ preserves compactness and commutes with arbitrary coproducts. The category $\mathscr{M}_{htp}$ is obtained from $\mathscr{M}_{str}$ by identifying homotopic DG functors. Finally, the category $\mathscr{M}$ is the localization of $\mathscr{M}_{htp}$ at the class of DG functors inducing equivalences in the stable categories. As for $\mathscr{M}^b$, this class admits a calculus of right fractions by Section 3.6.

An exact DG category $\mathscr{A}$ is *stably Karoubian* if $\underline{\mathscr{A}}$ is Karoubian. If $\mathscr{A}$ is a DG category in $\mathscr{M}^b$, then $\mathscr{A}^+$ is stably Karoubian since $\underline{\mathscr{A}^+}$ is compactly $\mathbf{U}$-generated. Clearly, there is a minimal Karoubian triangulated subcategory $\underline{\mathscr{A}}_{\mathrm{Kar}}$ of $\underline{\mathscr{A}^+}$ containing $\underline{\mathscr{A}}$. If we let $\mathscr{A}'_{\mathrm{Kar}}$ be the preimage of $\underline{\mathscr{A}}_{\mathrm{Kar}}$ in $\mathscr{A}^+$, then the stable category of $\mathscr{A}'_{\mathrm{Kar}}$ is clearly isomorphic to $\underline{\mathscr{A}}_{\mathrm{Kar}}$. However, $\mathscr{A}'_{\mathrm{Kar}}$ will not be $\mathbf{U}$-small in general. Therefore, define $\mathscr{A}_{\mathrm{Kar}}$ to be a minimal full subcategory of $\mathscr{A}'_{\mathrm{Kar}}$ containing $\mathscr{A}$ such that

(a) $\mathscr{A}_{\mathrm{Kar}}$ is closed under shifts and mapping cones,

(b) for each $A\in\mathscr{A}_{\mathrm{Kar}}$ and each idempotent e of $\underline{\mathscr{A}^+}(A,A)$, $\mathscr{A}_{\mathrm{Kar}}$ contains objects A' and A'' which in $\underline{\mathscr{A}^+}$ become isomorphic to the kernel and the image of e.

Then clearly $\mathscr{A}_{\mathrm{Kar}}$ is $\mathbf{U}$-small and its stable category is still equivalent to $\underline{\mathscr{A}}_{\mathrm{Kar}}$. In particular, it is stably Karoubian. By the theorem of Neeman–Ravenel (Section 4.12), $\underline{\mathscr{A}_{\mathrm{Kar}}}$ is in fact equivalent to the subcategory of compact objects of $\underline{\mathscr{A}^+}$.

Let $\mathscr{M}^b_{\mathrm{Kar}}$ be the full subcategory of $\mathscr{M}^b$ whose objects are the stably Karoubian categories.

Proposition. *The functor*

$$\mathscr{M}^b\to\mathscr{M},\quad \mathscr{A}\mapsto\mathscr{A}^+$$

admits a fully faithful right adjoint $\mathscr{B}\mapsto\mathscr{B}^c$. *The functors* $\mathscr{A}\mapsto\mathscr{A}^+$ *and* $\mathscr{B}\mapsto\mathscr{B}^c$ *induce quasi-inverse equivalences* $\mathscr{M}^b_{\mathrm{Kar}}\xrightarrow{\sim}\mathscr{M}$.

The proposition shows that $F^+ : \mathscr{A}^+ \to \mathscr{B}^+$ is invertible in $\mathscr{M}$ iff F is an equivalence up to factors. For $\mathscr{A}$ in $\mathscr{M}^b$, we have a canonical isomorphism $\mathscr{A}_{\mathsf{Kar}} \xrightarrow{\sim} (\mathscr{A}^+)^c$. In particular, we have an equivalence up to factors

$$\underline{\mathscr{A}} \to \underline{(\mathscr{A}^+)^c}.$$

For $\mathscr{B} \in \mathscr{M}$, we define the mixed complex

$$C(\mathscr{B}) := C(\mathscr{B}^c).$$

This means that $C(\mathscr{B}) = C(\mathscr{S})$ for some $\mathbf{U}$-small stably Karoubian DG subcategory $\mathscr{S} \subset \mathscr{B}$ containing a set of compact $\mathbf{U}$-generators for $\underline{\mathscr{B}}$. Since for $\mathscr{A} \in \mathscr{M}^b$, the canonical morphism $\mathscr{A} \to \mathscr{A}_{\mathsf{Kar}}$ induces an isomorphism $C(\mathscr{A}) \to C(\mathscr{A}_{\mathsf{Kar}})$, we have an isomorphism

$$C(\mathscr{A}) \xrightarrow{\sim} C(\mathscr{A}^+)$$

which is functorial in $\mathscr{A} \in \mathscr{M}^b$.

Proof of the proposition. We construct the right adjoint $\mathscr{B} \mapsto \mathscr{B}^c$. Let $\mathscr{B}$ be an object of $\mathscr{M}$. Consider the set $Cp(\mathscr{B})$ of stably Karoubian exact DG subcategories $\mathscr{R}$ of $\mathscr{B}$ such that $\mathscr{R}$ is $\mathbf{U}$-small and contains a family of $\mathbf{U}$-generators for $\underline{\mathscr{B}}$. The set $Cp(\mathscr{B})$ is non-empty and if $\mathscr{R}, \mathscr{R}'$ belong to $Cp(\mathscr{B})$ there is $\mathscr{R}'' \in Cp(\mathscr{B})$ containing both $\mathscr{R}$ and $\mathscr{R}'$. By the theorem of Neeman–Ravenel (Section 4.12), the inclusion functors then yield equivalences

$$\underline{\mathscr{R}} \xrightarrow{\sim} \underline{\mathscr{R}''} \xleftarrow{\sim} \underline{\mathscr{R}'}.$$

We put $\mathscr{B}^c = \mathscr{R}$ for some $\mathscr{R} \in Cp(\mathscr{B})$. This is independent of the choice of $\mathscr{R}$ up to canonical isomorphism. Now let $F : \mathscr{B}_1 \to \mathscr{B}_2$ be a functor such that $F : \underline{\mathscr{B}_1} \to \underline{\mathscr{B}_2}$ preserves compactness and commutes with arbitrary coproducts. We choose $\mathscr{R}_1 \in Cp(\mathscr{B}_1)$ arbitrarily and $\mathscr{R}_2 \in Cp(\mathscr{B}_2)$ such that $F\mathscr{R}_1 \subset \mathscr{R}_2$. This yields a well-defined morphism $F^c : \mathscr{B}_1^c \to \mathscr{B}_2^c$ of $\mathscr{M}$ and it is easy to check that $\mathscr{B} \mapsto \mathscr{B}^c$ is a functor. We will now construct a natural transformation $\mathscr{B} \to \mathscr{B}^{c+}$. By Section 3.6, we may and will assume that $\mathscr{B}$ (and hence $\mathscr{B}^c = \mathscr{R} \subset \mathscr{B}$) is a closed DG category. Let I be the $\mathscr{R}$–$\mathscr{R}$-bimodule $(X, Y) \mapsto \mathscr{R}(X, Y)$ and $\tilde{I} \to I$ a closed resolution over $\mathscr{R} \otimes \mathscr{R}$ such that $\tilde{I}(X, Y) \in \mathbf{U}$ for all $X, Y \in \mathscr{R}$. Since $\mathscr{R}$ is closed, the DG module $M \otimes_{\mathscr{R}} \tilde{I}$ is closed for each $\mathscr{R}$-module M such that $M(X)$ is a closed DG k-module for all $X \in \mathscr{R}$. In particular, we have a well-defined DG functor

$$\mathscr{B} \to \mathscr{R}^+, \quad B \mapsto \mathscr{B}(?, B) \otimes_{\mathscr{R}} \tilde{I}.$$

The associated functor in the stable categories commutes with arbitrary coproducts and induces an equivalence of $\underline{\mathscr{R}}$ onto its image in $\underline{\mathscr{R}^+}$. So this functor is indeed an equivalence $\underline{\mathscr{B}} \to \underline{\mathscr{R}^+}$. If $F : \mathscr{B}_1 \to \mathscr{B}_2$ is a DG functor as above, then we have to show that the diagram

$$\begin{array}{ccc} \mathscr{B}_1 & \xrightarrow{F} & \mathscr{B}_2 \\ \downarrow & & \downarrow \\ \underline{\mathscr{R}_1^+} & \xrightarrow{(F^c)^+} & \underline{\mathscr{R}_2^+} \end{array}$$

commutes in $\mathscr{M}$. We will even show that it commutes up to homotopy. Indeed with the obvious notations, the canonical morphism $\tilde{I}_2 \to I_2$ yields an isomorphism

$$\mathscr{B}_1(?,X) \otimes_{\mathscr{R}_1} \tilde{I}_1 \otimes_{\mathscr{R}_1} \mathscr{R}_2(-,F?) \otimes_{\mathscr{R}_2} \tilde{I}_2 \to \mathscr{B}_1(?,X) \otimes_{\mathscr{R}_1} \tilde{I}_1 \otimes_{\mathscr{R}_1} \mathscr{R}_2(-,F?)$$

of $\underline{\mathscr{R}_2^+}$, for each $X \in \mathscr{B}_1$. Now consider the composition

$$\begin{aligned} \mathscr{B}_1(?,X) \otimes_{\mathscr{R}_1} \tilde{I}_1 \otimes_{\mathscr{R}_1} \mathscr{R}_2(-,F?) \otimes_{\mathscr{R}_2} \tilde{I}_2 &\to \mathscr{B}_1(?,X) \otimes_{\mathscr{R}_1} \mathscr{R}_2(-,F?) \otimes_{\mathscr{R}_2} \tilde{I}_2 \\ &\to \mathscr{B}_2(-,FX) \otimes_{\mathscr{R}_2} \tilde{I}_2, \end{aligned}$$

where the first morphism is induced by the canonical morphism $\tilde{I}_1 \to I_1$ and the second by the canonical morphism

$$\mathscr{B}_1(?,X) \otimes \mathscr{R}_2(-,F?) \to \mathscr{B}_2(-,FX).$$

The above composition is clearly invertible in $\underline{\mathscr{R}_2^+}$ for $X \in \mathscr{R}_1$. By infinite dévissage, this suffices to conclude that it is an isomorphism for arbitrary $X \in \mathscr{B}_1$.

Finally, for $\mathscr{A} \in \mathscr{M}_{\mathrm{Kar}}$, we have a canonical isomorphism $\mathscr{A} \to (\mathscr{A}^+)^c$ in $\mathscr{M}$. Indeed, we can take for $\mathscr{R} \in Cp(\mathscr{A}^+)$ the image of the Yoneda embedding $A \mapsto \mathscr{A}(?,A)$. This is clearly natural in $\mathscr{A} \in \mathscr{M}_{\mathrm{Kar}}$. □

4.2. Exact sequences in $\mathscr{M}^b$ and $\mathscr{M}$

Let

$$\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C}$$

be a sequence of $\mathscr{M}^b$.

Lemma. *The induced sequence*

$$0 \to \underline{\mathscr{A}^+} \to \underline{\mathscr{B}^+} \to \underline{\mathscr{C}^+} \to 0$$

is exact in $\mathscr{M}$ if and only if the sequence of triangulated categories

$$0 \to \underline{\mathscr{A}} \to \underline{\mathscr{B}} \to \underline{\mathscr{C}} \to 0$$

is exact up to factors.

Proof. By the theorem of Neeman–Ravenel (Section 4.12), the sequence

$$0 \to \underline{\mathscr{A}^+} \to \underline{\mathscr{B}^+} \to \underline{\mathscr{C}^+} \to 0$$

is exact iff the subcategories of compact objects form a sequence which is exact up to factors. Now we know that the subcategory of compact objects of $\underline{\mathscr{A}^+}$ is equivalent to $\underline{\mathscr{A}_{\mathrm{Kar}}}$. Clearly, exactness up to factors is preserved by passage between $\underline{\mathscr{A}}$ and $\underline{\mathscr{A}_{\mathrm{Kar}}}$. The claim follows. □

4.3. *The categories $\mathscr{L}^b$ and $\mathscr{L}$*

This is a relative version of Section 4.1. Let $\mathscr{L}^b_{str}$ denote the category whose objects are the localization pairs $\mathscr{A}_0 \subset \mathscr{A}_1$ with U-small $\mathscr{A}_1$. Morphisms are morphisms of localization pairs. The category $\mathscr{L}^b_{htp}$ is obtained by identifying homotopic morphisms and the category $\mathscr{L}^b$ is obtained from $\mathscr{L}^b_{htp}$ by localization at the class of morphisms $F: \mathscr{A} \to \mathscr{B}$ inducing equivalences $\underline{\mathscr{A}_i} \xrightarrow{\sim} \underline{\mathscr{B}_i}$, $i = 0, 1$. In Section 3.9, we have shown that this class admits a calculus of right fractions and we have constructed the functor

$$Cm : \mathscr{L}^b \to \mathscr{D}\mathscr{M}or\mathscr{M}ix, \quad \mathscr{A} \mapsto Cm(\mathscr{A}).$$

Let $\mathscr{L}_{str}$ be the category whose objects are pairs $\mathscr{A} : \mathscr{A}_0 \subset \mathscr{A}_1$ of exact DG categories belonging to $\mathscr{M}$ such that $\mathscr{A}_0$ is *saturated*, i.e. its image in $\underline{\mathscr{A}_1}$ is closed under isomorphism. By definition the morphisms $\mathscr{A} \to \mathscr{A}'$ of $\mathscr{L}_{str}$ are DG functors $F : \mathscr{A}_1 \to \mathscr{A}'_1$ of $\mathscr{M}_{str}$ such that $F\mathscr{A}_0 \subset \mathscr{A}'_0$. The category $\mathscr{L}_{htp}$ is obtained from $\mathscr{L}_{str}$ by identifying homotopic functors. The category $\mathscr{L}$ is deduced from $\mathscr{L}_{htp}$ by localizing at the class of functors inducing equivalences in the associated stable categories. This class admits a calculus of right fractions by Lemma 3.9.

For a localization pair $\mathscr{A} = (\mathscr{A}_0, \mathscr{A}_1)$, we let $\mathscr{A}^+ \in \mathscr{L}$ be the pair consisting of $\mathscr{A}_1^+$ and the saturation of the image of $\mathscr{A}_0^+$ in $\mathscr{A}_1^+$. This yields a functor $\mathscr{L}^b \to \mathscr{L}$.

A pair $\mathscr{A} = (\mathscr{A}_0, \mathscr{A}_1)$ of $\mathscr{L}^b$ is *stably Karoubian* if both $\mathscr{A}_0$ and $\mathscr{A}_1$ are stably Karoubian. In analogy with Section 4.1, one can construct a canonical morphism $\mathscr{A} \to \mathscr{A}_{\mathrm{Kar}}$ to a stably Karoubian pair $\mathscr{A}_{\mathrm{Kar}}$. The canonical morphism $\mathscr{A}^+ \to (\mathscr{A}_{\mathrm{Kar}})^+$ is an isomorphism. The following proposition now shows that $\mathscr{A} \to \mathscr{A}_{\mathrm{Kar}}$ is universal among the morphisms from $\mathscr{A}$ to an object of $\mathscr{L}^b_{\mathrm{Kar}}$, the full subcategory of $\mathscr{L}^b$ whose objects are the stable Karoubian localization pairs.

Proposition. *The functor*

$$\mathscr{L}^b \to \mathscr{L}, \ \mathscr{A} \to \mathscr{A}^+$$

induces an equivalence $\mathscr{L}^b_{\mathrm{Kar}} \xrightarrow{\sim} \mathscr{L}$.

The proof of the proposition is a variation on the proof of Section 4.1. Let $\mathscr{B} \mapsto \mathscr{B}^c$ denote an quasi-inverse functor. For $\mathscr{B} \in \mathscr{L}$, we define the object

$$Cm(\mathscr{B}) := Cm(\mathscr{B}^c).$$

This means that $Cm(\mathscr{B}) = Cm(\mathscr{S})$ for a localization pair $(\mathscr{S}_0, \mathscr{S}_1)$ such that $\mathscr{S}_i \subset \mathscr{B}_i$ is a U-small stably Karoubian DG subcategory containing a set of compact U-generators for $\underline{\mathscr{B}_i}$, $i=1,2$. As in Section 4.1, we have an isomorphism

$$Cm(\mathscr{A}) \xrightarrow{\sim} Cm(\mathscr{A}^+)$$

which is functorial in $\mathscr{A} \in \mathscr{L}^b$.

4.4. *Exact sequences of $\mathscr{M}_{str}$*

As in Section 4.1 let $\mathbf{U}$ be a universe containing an infinite set and k a commutative ring in $\mathbf{U}$. Let the sequence

$$0 \to \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \to 0$$

of $\mathscr{M}_{str}$ be *exact*, i.e. the following conditions hold:

(a) We have $GF=0$.

(b) The functor F admits a right adjoint DG functor F_ρ such that the adjunction morphism $\mathbf{1}_{\mathscr{A}} \to F_\rho F$ is invertible.

(c) The functor G admits a right adjoint DG functor G_ρ such that the adjunction morphism $GG_\rho \to \mathbf{1}_{\mathscr{C}}$ is invertible.

(d) For each $B \in \mathscr{B}$, the sequence

$$FF_\rho B \to B \to G_\rho GB$$

is a conflation of $\mathscr{B}$.

Note that the adjoint functors F_ρ and G_ρ are *not* required to be morphisms of $\mathscr{M}_{str}$. In general, they will induce functors in the stable categories which do not preserve compactness. However, they commute with arbitrary coproducts by the following lemma.

Lemma. *Let $\mathscr{S}$ be a compactly generated triangulated category, $\mathscr{T}$ a triangulated category with arbitrary coproducts and $F:\mathscr{S}\to\mathscr{T}$ a triangle functor preserving compactness and commuting with arbitrary coproducts. If F_ρ is right adjoint to F, then F_ρ commutes with arbitrary coproducts. Moreover F detects compactness, i.e. an object X of $\mathscr{S}$ is compact iff so is FX.*

Proof. Let $A \in \mathscr{S}$ be small and B_i, $i \in I$, a family of $\mathscr{T}$ with $I \in \mathbf{U}$. Using the compactness of FA and A we obtain the following chain of isomorphisms

$$\mathscr{S}\left(A, F_\rho\left(\coprod_{i\in I} B_i\right)\right) \xrightarrow{\sim} \mathscr{T}\left(FA, \coprod_{i\in I} B_i\right) \xrightarrow{\sim} \coprod_{i\in I} \mathscr{T}(FA, B_i)$$
$$\xrightarrow{\sim} \coprod_{i\in I} \mathscr{T}(A, F_\rho B_i) \xrightarrow{\sim} \mathscr{S}\left(A, \coprod_{i\in I} F_\rho B_i\right).$$

Since $\mathscr{S}$ is compactly generated, it follows that F_ρ commutes with coproducts. The second assertion is immediate from the faithfulness of F and the fact that it commutes with arbitrary coproducts. $\square$

4.5. Exact sequences of $\mathcal{M}_{htp}$

By definition, an *exact sequence* of $\mathcal{M}_{htp}$ is a sequence satisfying the hypothesis of part (a) of the following lemma. Note that each exact sequence of $\mathcal{M}_{str}$ yields an exact sequence of $\mathcal{M}_{htp}$.

Lemma. (a) *Let $\mathcal{A}' \xrightarrow{F'} \mathcal{B}' \xrightarrow{G'} \mathcal{C}'$ be a sequence of $\mathcal{M}_{htp}$ such that the induced sequence of triangulated categories*

$$0 \to \underline{\mathcal{A}'} \to \underline{\mathcal{B}'} \to \underline{\mathcal{C}'} \to 0$$

is exact. Then there is an exact sequence

$$0 \to \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \to 0$$

of $\mathcal{M}_{str}$ and a commutative diagram of $\mathcal{M}_{htp}$

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{A}' \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{C}' \end{tikzcd}$$

such that the vertical arrows induce equivalences in the associated stable categories.

(b) *Let $\mathcal{A}' \xrightarrow{F'} \mathcal{B}'$ be a morphism of $\mathcal{M}_{htp}$ inducing a fully faithful functor $\underline{\mathcal{A}'} \to \underline{\mathcal{B}'}$. Then there is an exact sequence (F', G') of $\mathcal{M}_{htp}$.*

Proof. (a) The proof is the one of [19, 6.1]. For completeness, we give the construction: Let $\mathcal{A} = \mathcal{A}'$ and let $\mathcal{B}$ be the full subcategory of $\operatorname{Fil} F'$ whose objects are the pairs $(A, i : F'A \to B)$ such that we have

$$\underline{\mathcal{B}'}(FA, \operatorname{cok} i) = 0$$

for all $A \in \mathcal{A}$. Let $\mathcal{C}$ be the subcategory of $\mathcal{B}$ whose objects are the pairs with $A = 0$. We have the functors

$$\begin{aligned} &\mathcal{A} \to \mathcal{B}, && A \mapsto (A, \mathbf{1} : F'A \to F'A) \\ &\mathcal{B} \to \mathcal{C}, && (A, i : F'A \to B) \mapsto (0, 0 \to \operatorname{cok} i) \\ &\mathcal{B} \to \mathcal{B}', && (A, i : F'A \to B) \mapsto B \\ &\mathcal{C} \to \mathcal{C}', && (0, 0 \to C) \mapsto G'C. \end{aligned}$$

This construction also yields a proof of (b). $\square$

4.6. Exact sequences of $\mathcal{M}$

By definition, an *exact sequence* of $\mathcal{M}$ is a sequence satisfying the hypothesis of part (a) of the following:

Theorem. (a) *Let* $\mathscr{A}' \xrightarrow{F'} \mathscr{B}' \xrightarrow{G'} \mathscr{C}'$ *be a sequence of* $\mathscr{M}$ *such that the induced sequence of triangulated categories*

$$0 \to \underline{\mathscr{A}'} \to \underline{\mathscr{B}'} \to \underline{\mathscr{C}'} \to 0$$

is exact. Then the morphism F' *is a monomorphism of* $\mathscr{M}$, *the morphism* G' *is an epimorphism and the diagram*

$$\begin{tikzcd} \mathscr{A}' \arrow[r] \arrow[d] & \mathscr{B}' \arrow[d] \\ 0 \arrow[r] & \mathscr{C}' \end{tikzcd}$$

is a pullback and a pushout in $\mathscr{M}$.

(b) *Let* $\mathscr{A}' \xrightarrow{F'} \mathscr{B}'$ *be a morphism of* $\mathscr{M}$ *such that the induced functor* $\underline{\mathscr{A}'} \to \underline{\mathscr{B}'}$ *is fully faithful. Then there is an exact sequence* (F', G') *of* $\mathscr{M}$.

Proof. (a) Since $\mathscr{M}$ is obtained from $\mathscr{M}_{htp}$ by localizing at a class admitting a calculus of right fractions, and by Lemma 4.5, we may assume that we have in fact an exact sequence of $\mathscr{M}_{str}$

$$0 \to \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \to 0.$$

Again by the calculus of right fractions, to prove that F is a monomorphism and that the above square is cartesian, it is enough to prove the corresponding assertions in $\mathscr{M}_{htp}$. Let $\mathscr{X} \in \mathscr{M}$. Recall the category $\mathrm{Fun}(\mathscr{X}, \mathscr{A})$ from Example 2.2(e). Let $\mathrm{fun}(\mathscr{X}, \mathscr{A})$ denote the subcategory of functors $H \in \mathrm{Fun}(\mathscr{X}, \mathscr{A})$ such that the induced functor in the stable categories preserves compactness and commutes with arbitrary coproducts. Denote by Σ the class of morphisms s of the stable category of $\mathrm{Fun}(\mathscr{X}, \mathscr{A})$ such that sX is invertible in $\underline{\mathscr{A}}$ for all $X \in \mathscr{X}$. Clearly this system is compatible with the triangulated structure and if H is in $\mathrm{fun}(\mathscr{X}, \mathscr{A})$ and $s : H \to H'$ is a morphism of Σ then H' belongs to $\mathrm{fun}(\mathscr{X}, \mathscr{A})$. Let $\mathrm{rep}(\mathscr{X}, \mathscr{A})$ denote the localization of the stable category of $\mathrm{fun}(\mathscr{X}, \mathscr{A})$ at the class of morphisms of Σ between objects of $\mathrm{fun}(\mathscr{X}, \mathscr{A})$. By what we have just seen, $\mathrm{rep}(\mathscr{X}, \mathscr{A})$ identifies with a full subcategory of $\mathrm{Rep}(\mathscr{X}, \mathscr{A})$, the localization of the stable category of $\mathrm{Fun}(\mathscr{X}, \mathscr{A})$ at Σ. Consider the sequences

$$\begin{tikzcd} \mathrm{Fun}(\mathscr{X}, \mathscr{A}) \arrow[r, "F_*"] & \mathrm{Fun}(\mathscr{X}, \mathscr{B}) \arrow[r, "G_*"] & \mathrm{Fun}(\mathscr{X}, \mathscr{C}) \\ \mathrm{fun}(\mathscr{X}, \mathscr{A}) \arrow[r] \arrow[u] & \mathrm{fun}(\mathscr{X}, \mathscr{B}) \arrow[r] \arrow[u] & \mathrm{fun}(\mathscr{X}, \mathscr{C}). \arrow[u] \end{tikzcd}$$

The functors F_* and $(F_\rho)_*$ are a pair of adjoint functors, they are compatible with Σ and the composition $(F_\rho)_* F_*$ is isomorphic to the identity. Hence F_* induces a fully faithful functor $\mathrm{Rep}(\mathscr{X}, \mathscr{A}) \to \mathrm{Rep}(\mathscr{X}, \mathscr{B})$. Moreover F_* takes $\mathrm{rep}(\mathscr{X}, \mathscr{A})$ to $\mathrm{rep}(\mathscr{X}, \mathscr{B})$.

So it induces a fully faithful functor $\mathrm{rep}(\mathscr{X},\mathscr{A}) \to \mathrm{rep}(\mathscr{X},\mathscr{B})$. Now by definition, the morphisms from $\mathscr{X}$ to $\mathscr{A}$ of $\mathscr{M}_{htp}$ are the isomorphism classes of functors in $\mathrm{rep}(\mathscr{X},\mathscr{A})$. So the map induced by F on the sets of morphisms of $\mathscr{M}_{htp}$ is injective and F is a monomorphism of $\mathscr{M}_{htp}$. Now suppose that we have $H \in \mathrm{rep}(\mathscr{X},\mathscr{B})$ such that $GH = 0$ in $\mathscr{M}_{htp}$. Then by the triangles

$$FF_\rho HX \to HX \to G_\rho GHX \to SFF_\rho HX, \quad X \in \mathscr{X},$$

H becomes isomorphic to $FF_\rho H$ in $\mathrm{rep}(\mathscr{X},\mathscr{B})$. Since $\underline{F}$ detects compactness (Section 4.4) and $\underline{F_\rho}$ commutes with arbitrary coproducts it follows that $F_\rho H$ belongs to the category $\mathrm{rep}(\mathscr{X},\mathscr{A})$. So the square of the assertion is a pullback.

We will now show that G is an epimorphism of $\mathscr{M}_{htp}$ and that the square is a pushout in $\mathscr{M}_{htp}$. Indeed, consider the sequences

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{C},\mathscr{X}) \arrow[r, "G^*"] & \mathrm{Fun}(\mathscr{B},\mathscr{X}) \arrow[r] & \mathrm{Fun}(\mathscr{A},\mathscr{X}) \\
\mathrm{fun}(\mathscr{C},\mathscr{X}) \arrow[r, "G^*"] \arrow[u] & \mathrm{fun}(\mathscr{B},\mathscr{X}) \arrow[r] \arrow[u] & \mathrm{fun}(\mathscr{A},\mathscr{X}) \arrow[u]
\end{tikzcd}$$

As above, we see that G^* induces a fully faithful functor $\mathrm{rep}(\mathscr{C},\mathscr{X}) \to \mathrm{rep}(\mathscr{B},\mathscr{X})$. Now suppose that we have $H : \mathscr{B} \to \mathscr{X}$ such that $HF = 0$ in $\mathscr{M}_{htp}$. Then as above we see that H is isomorphic to $HG_\rho G$ in $\mathrm{rep}(\mathscr{B},\mathscr{X})$. We claim that HG_ρ belongs to $\mathrm{rep}(\mathscr{C},\mathscr{X})$. Indeed, the functor induced by G_ρ in the stable categories commutes with arbitrary coproducts. Hence so does HG_ρ. Moreover, we know from the theorem of Neeman–Ravenel (4.12) that the compact objects of $\underline{\mathscr{C}}$ are direct factors of objects GB, where B is compact in $\underline{\mathscr{B}}$. For such an object, $HG_\rho(GB) \xrightarrow{\sim} HB$ is compact by assumption. Whence the claim.

The fact that G is an epimorphism of $\mathscr{M}$ and that the square is a pushout in $\mathscr{M}$ is now easily deduced from the calculus of fractions and Lemma 4.5.

(b) results from Section 4.5(b) by the calculus of fractions. □

4.7. Comparison of $\mathscr{L}$ and short exact sequences of $\mathscr{M}$

Denote by Ex the category of exact sequences of $\mathscr{M}$. Let $\mathscr{A} = (\mathscr{A}_0, \mathscr{A}_1)$ be an object of $\mathscr{L}$. The inclusion functor $\underline{\mathscr{A}_0} \to \underline{\mathscr{A}_1}$ is fully faithful so that by Section 4.6 we have an exact sequence

$$0 \to \mathscr{A}_0 \to \mathscr{A}_1 \to \mathscr{A}_2 \to 0$$

of $\mathscr{M}$, where $\mathscr{A}_2$ is unique up to unique isomorphism in $\mathscr{M}$. We thus obtain a functor

$$\Phi : \mathscr{L} \to \mathrm{Ex}\,.$$

On the other hand, suppose that

$$\varepsilon : 0 \to \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \to 0$$

is an exact sequence of $\mathcal{M}$. We define $\Psi(\varepsilon)$ to be the pair consisting of $\mathcal{B}$ and the saturation of the image of $F\mathcal{A}$ in $\mathcal{B}$.

Lemma. *The functors Φ and Ψ are quasi-inverse equivalences.*

Proof. Let Adm be the category of admissible monos of $\mathcal{M}$, i.e. of morphisms $F:\mathcal{A} \to \mathcal{B}$ of $\mathcal{M}$ such that the functor $\underline{F}:\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ is fully faithful. By Section 4.6, the canonical functor Ex $\to$ Adm is an equivalence. So it is enough to show that

$$\Phi' : \mathcal{L} \to \mathrm{Adm}\,, \quad (\mathcal{A}_0, \mathcal{A}_1) \mapsto (\mathcal{A}_0 \to \mathcal{A}_1)$$

is an equivalence whose quasi-inverse functor is Ψ': Adm $\to \mathcal{L}$ defined as follows: By definition the image of $F:\mathcal{A} \to \mathcal{B}$ under Ψ is the pair formed by $\mathcal{B}_1 = \mathcal{B}$ and the saturation $\mathcal{B}_0$ of the image of F. Clearly we have $\Psi'\Phi' = \mathbf{1}$. On the other hand, we have a canonical morphism $\Phi'\Psi' \to \mathbf{1}$ given for $F:\mathcal{A} \to \mathcal{B}$ by the square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\mathrm{incl}"'] & \mathcal{B} \arrow[d, "\mathbf{1}"] \\ \mathcal{B}_0 \arrow[r] & \mathcal{B}_0. \end{tikzcd}$$

This morphism is clearly functorial and invertible in Adm. $\square$

4.8. Cokernels

Consider the functor

$$\mathcal{M} \to \mathrm{Ex}\,, \quad \mathcal{A} \to (0 \to \mathcal{A} \xrightarrow{\mathbf{1}} \mathcal{A}).$$

By Section 4.6, this functor admits a left adjoint. By Section 4.7, it follows that the functor

$$I : \mathcal{M} \to \mathcal{L}, \;\; \mathcal{A} \to (\mathrm{Sat}\,(0), \mathcal{A})$$

admits a left adjoint, where Sat (0) is the saturation of the zero subcategory, i.e. the subcategory of injectives of $\mathcal{A}$. We denote the left adjoint by I_λ. If $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ is an object of $\mathcal{L}$, we also write $\mathcal{A}_1/\mathcal{A}_0$ for $I_\lambda\mathcal{A}$. Note that by definition, we have an exact sequence

$$0 \to \mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_1/\mathcal{A}_0 \to 0$$

of $\mathcal{M}$.

Now suppose that $\mathcal{A} = \mathcal{B}^+$ for a localization pair $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$. Put $Q^+\mathcal{B} = I_\lambda\mathcal{B}^+$. Then we have an exact sequence of $\mathcal{M}$

$$0 \to \mathcal{B}_0^+ \to \mathcal{B}_1^+ \to Q^+\mathcal{B} \to 0$$

yielding an exact sequence of triangulated categories

$$0 \to \underline{\mathscr{B}_0^+} \to \underline{\mathscr{B}_1^+} \to \underline{Q^+\mathscr{B}} \to 0$$

By the theorem of Neeman–Ravenel (Section 4.12), if we pass to the subcategories of compact objects, we obtain a sequence which is exact up to factors

$$0 \to \underline{\mathscr{B}_0^{+}}^c \to \underline{\mathscr{B}_1^{+}}^c \to \underline{Q^+\mathscr{B}}^c \to 0$$

Now for each object $\mathscr{A}$ of $\mathscr{M}^b$, the canonical functor $\underline{\mathscr{A}} \to \underline{\mathscr{A}^{+}}^c$ is an equivalence up to factors (Section 4.1). Hence we have a diagram of triangulated categories

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathscr{B}_0^{+}}^c \arrow[r] & \underline{\mathscr{B}_1^{+}}^c \arrow[r] & \underline{Q^+\mathscr{B}}^c \arrow[r] & 0 \\
0 \arrow[r] & \underline{\mathscr{B}_0} \arrow[r] \arrow[u] & \underline{\mathscr{B}_1} \arrow[r] \arrow[u] & \underline{\mathscr{B}_1}/\underline{\mathscr{B}_0} \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

The first two vertical functors are equivalences up to factors and hence (exercise) so is the third. Recall that $\mathscr{T}\mathscr{B} = \underline{\mathscr{B}_1}/\underline{\mathscr{B}_0}$ by definition. So we have an equivalence up to factors

$$\mathscr{T}\mathscr{B} \to \underline{Q^+\mathscr{B}}^c.$$

By construction, this equivalence is functorial in $\mathscr{B}$, up to isomorphism of triangulated functors. In particular, if $F : \mathscr{B} \to \mathscr{B}'$ induces an equivalence up to factors $\mathscr{T}\mathscr{B} \to \mathscr{T}\mathscr{B}'$, then it induces an isomorphism of $\mathscr{M}$

$$Q^+\mathscr{B} \to Q^+\mathscr{B}'$$

by Corollary 4.12(c) and if

$$\mathscr{B}' \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{B}''$$

yields a sequence

$$0 \to \mathscr{T}\mathscr{B}' \to \mathscr{T}\mathscr{B} \to \mathscr{T}\mathscr{B}'' \to 0$$

which is exact up to factors, then by Corollary 4.12 (b), the sequence (F, G) induces an exact sequence

$$0 \to Q^+\mathscr{B}' \to Q^+\mathscr{B} \to Q^+\mathscr{B}'' \to 0$$

of $\mathscr{M}$.

4.9. Cokernels and cones

If

$$\varepsilon : 0 \to \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \to 0$$

is an exact sequence of $\mathscr{M}$, we put, in the notations of Section 4.7,

$$Cm(\varepsilon) := Cm(\Psi(\varepsilon)).$$

For each object

$$X = (X_1 \to X_2) \in \mathscr{D}\mathscr{M}or\mathscr{M}ix,$$

we define ∂X to be the connecting morphism of the canonical triangle

$$X_0 \xrightarrow{f} X_1 \to \mathrm{Cone}(X) \xrightarrow{\partial X} X_1[1].$$

Theorem. (a) *The square*

$$\begin{tikzcd} \mathscr{L} \arrow[r, "Cm"] \arrow[d, "l_\lambda"'] & \mathscr{D}\mathscr{M}or\mathscr{M}ix \arrow[d, "\mathrm{Cone}"] \\ \mathscr{M} \arrow[r, "C"] & \mathscr{D}\mathscr{M}ix \end{tikzcd}$$

is commutative up to natural isomorphism.

(b) *For each exact sequence ε of $\mathscr{M}$, there is a commutative diagram of $\mathscr{D}\mathscr{M}ix$*

$$\begin{tikzcd} Cm(\varepsilon)_1 \arrow[r] \arrow[d] & Cm(\varepsilon)_2 \arrow[r] \arrow[d] & \mathrm{Cone}(Cm(\varepsilon)) \arrow[d] \\ C(\mathscr{A}) \arrow[r] & c(\mathscr{B}) \arrow[r] & C(\mathscr{C}) \end{tikzcd}$$

whose vertical morphisms are invertible. This diagram is functorial in ε. In particular, if we define $\partial\varepsilon$ by the commutative square

$$\begin{tikzcd} \mathrm{Cone}(Cm(\varepsilon)) \arrow[r, "\partial Cm(\varepsilon)"] \arrow[d] & Cm(\varepsilon)_1 \arrow[d] \\ C(\mathscr{A}) \arrow[r, "\partial\varepsilon"] & C(\mathscr{A})[1] \end{tikzcd}$$

then we have a functorial triangle

$$C(\mathscr{A}) \to C(\mathscr{B}) \to C(\mathscr{C}) \xrightarrow{\partial\varepsilon} C(\mathscr{A})[1].$$

Proof. For a pair $(\mathscr{A}_0, \mathscr{A}_1)$, we have the adjunction morphism $\varphi : (\mathscr{A}_0, \mathscr{A}_1) \to (\mathrm{Sat}(0), \mathscr{A}_1/\mathscr{A}_0)$. We therefore obtain a natural morphism

$$\mathrm{Cone}(Cm(\mathscr{A}_0, \mathscr{A}_1)) \xrightarrow{\omega} \mathrm{Cone}(Cm(\mathrm{Sat}(0), \mathscr{A}_1/\mathscr{A}_0)) \xrightarrow{\sim} C(\mathscr{A}_1/\mathscr{A}_0).$$

We will show that the morphism $\omega = \mathrm{Cone}(Cm(\varphi))$ is invertible. This will show (a). By the equivalence $\mathscr{L} \xrightarrow{\sim} \mathrm{Ex}$, we may assume that we have an exact sequence

$$0 \to \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \to 0$$

of $\mathcal{M}_{str}$ such that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are closed DG categories and and $\mathcal{A}_1 = \mathcal{B}$ and $\mathcal{A}_0 = \mathrm{Sat}\,(F\mathcal{A})$. Then the adjunction morphism is induced by the functor G and is given by the square

$$\begin{tikzcd} \mathrm{Sat}(F\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "G."] \\ \mathrm{Sat}(0) \arrow[r] & \mathcal{C} \end{tikzcd}$$

To compute the image of this morphism under Cone $\circ\, Cm$, we have to choose suitable U-small subcategories. In the notations of the proof of Proposition 4.1 let $\mathcal{S} \in Cp(\mathcal{B})$ and $\mathcal{R} \in Cp(\mathcal{A})$ such that $F\mathcal{R} \subset \mathcal{S}$. Consider the square

$$\begin{tikzcd} C(F\mathcal{R}) \arrow[r] \arrow[d] & C(\mathcal{S}) \arrow[d, "C(G)"] \\ C(0) \arrow[r, "\mathrm{incl}"'] & C(\mathcal{T}) \end{tikzcd}$$

If we interpret its rows as objects of $\mathcal{DM}or\mathcal{M}ix$, it represents the image of the adjunction morphism under the functor Cm. Now the functor F is fully faithful, so we have an isomorphism $C(\mathcal{R}) \xrightarrow{\sim} C(F\mathcal{R})$. Hence we are reduced to showing that the square

$$\begin{tikzcd} C(\mathcal{R}) \arrow[r, "C(F)"] \arrow[d] & C(\mathcal{S}) \arrow[d, "C(G)"] \\ 0 \arrow[r] & C(\mathcal{T}) \end{tikzcd}$$

represents a morphism of $\mathcal{DM}or\mathcal{M}ix$ whose image under the Cone-functor is invertible in $\mathcal{DM}ix$. This is proved in Section 4.13. The above argument also proves (b). □

4.10. Mayer–Vietoris-diagrams

Let

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{B} \arrow[r, "G"] \arrow[d, "L"'] & \mathcal{C} \arrow[r] \arrow[d, "M"] & 0 \\ 0 \arrow[r] & \mathcal{A}' \arrow[r, "F'"] & \mathcal{B}' \arrow[r, "G'"] & \mathcal{C}' \arrow[r] & 0 \end{tikzcd}$$

be a diagram of $\mathcal{M}$ whose rows are exact such that K is invertible. We use the notations of Section 4.9.

Lemma. *If δ denotes the composition*

$$C(\mathscr{C}') \to \mathscr{C}(\mathscr{A}')[1] \xrightarrow{C(K)[1]^{-1}} C(\mathscr{A})[1] \to C(\mathscr{B})[1],$$

the sequence

$$C(\mathscr{B}) \xrightarrow{\alpha} C(\mathscr{C}) \oplus C(\mathscr{B}') \xrightarrow{\beta} C(\mathscr{C}') \xrightarrow{\delta C(\mathscr{B})[1]} C(\mathscr{B})[1], \qquad \alpha = \begin{bmatrix} -C(G) \\ C(L) \end{bmatrix},$$
$$\beta = [C(M), C(G')],$$

is a triangle of $\mathscr{D}\mathcal{M}ix$.

Proof. We are given a morphism $\varepsilon \to \varepsilon'$ of exact sequences of $\mathcal{M}$. It induces amorphism $Cm(\varepsilon) \to Cm(\varepsilon')$ of $\mathscr{D}\mathcal{M}or\mathcal{M}ix$ whose first component $Cm(\varepsilon)_1 \to Cm(\varepsilon')_1$ is invertible. After replacing $Cm(\varepsilon)$ and $Cm(\varepsilon')$ by isomorphic objects, we may assume that the morphism is given by a morphism

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d, "f"'] & X_1 \arrow[d, "g"] \\ X_0' \arrow[r, "i'"'] & X_1' \end{tikzcd}$$

of $\mathcal{M}or\mathcal{M}ix$, where i and i' are componentwise injective and f is a quasi-isomorphism. After replacing $X_0 \to X_1$ by

$$X_0' \to X_0' \oplus_{X_0} X_1$$

we may even assume that f is an isomorphism of $\mathcal{M}ix$. Then we have a diagram with exact rows in $\mathcal{M}ix$. By passing to the associated triangles in $\mathscr{D}\mathcal{M}ix$ we obtain the assertion. □

4.11. *Proofs of Theorems 2.4 and 2.7*

Let $\mathscr{B} = (\mathscr{B}_0, \mathscr{B}_1)$ be a localization pair. We have a natural isomorphism $Cm(\mathscr{B}) \xrightarrow{\sim} Cm(\mathscr{B}^+)$ by Section 4.3. This yields the first isomorphism in

$$\mathrm{Cone}(Cm(\mathscr{B})) \xrightarrow{\sim} \mathrm{Cone}(Cm(\mathscr{B}^+)) \xrightarrow{\sim} C(I_\lambda \mathscr{B}^+) = C(Q^+\mathscr{B}).$$

The second one is from Section 4.9(a). Now Theorem 2.4 follows from Sections 4.8 and 4.9(b), and Theorem 2.7 follows from Sections 4.8 and 4.10.

4.12. The Neeman–Ravenel–Thomason–Trobaugh–Yao theorem

Let $\mathcal{S}$ be a triangulated category admitting arbitrary set-indexed coproducts. An object X of $\mathcal{S}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{S}}(X, ?)$ commutes with arbitrary coproducts. The category $\mathcal{S}$ is *compactly generated* if it contains a set of compact objects C such that $\mathcal{T}$ coincides with its smallest triangulated subcategory containing C and stable under forming coproducts.

The following theorem is due to Neeman [29, 2.1]. His proof is based on ideas of Ravenel [31]. Important special cases are due to Thomason–Trobaugh [32] and Yao [39].

Theorem. *Let $\mathcal{R}$ and $\mathcal{S}$ be compactly generated triangulated categories. Suppose that R is a set of compact objects generating $\mathcal{R}$. Let $F : \mathcal{R} \to \mathcal{S}$ be a fully faithful functor commuting with arbitrary coproducts and such that FX is compact for each $X \in R$. Put $\mathcal{T} = \mathcal{S}/\mathcal{R}$.*

The functors $F : \mathcal{R} \to \mathcal{S}$ and $G : \mathcal{S} \to \mathcal{T}$ preserve compactness. The natural functor $\mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful and $\mathcal{T}^c$ is the closure of its image under forming direct factors.

The theorem (and its proof) admit the following

Corollary. (a) *Suppose that $\mathcal{S}$ is a triangulated category admitting arbitrary coproducts which is generated by a set R of compact objects. Then an object of $\mathcal{S}$ is compact iff it is a direct factor of a finite extension of objects of R.*

(b) *Suppose that $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ are compactly generated triangulated categories. Then a sequence*

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{T} \to 0$$

of triangle functors preserving compactness and commuting with coproducts is exact iff the induced sequence

$$0 \to \mathcal{R}^c \to \mathcal{S}^c \to \mathcal{T}^c \to 0$$

is exact up to factors of Section 2.4.

(c) *Suppose that $\mathcal{R}$ and $\mathcal{S}$ are compactly generated triangulated categories. Then a triangle functor $\mathcal{R} \to \mathcal{S}$ preserving compactness and commuting with arbitrary coproducts is an equivalence iff it induces an equivalence up to factors $\mathcal{R}^c \to \mathcal{S}^c$. This holds iff it induces an equivalence $\mathcal{R}^c \xrightarrow{\sim} \mathcal{S}^c$.*

Proof of the corollary. Part (a) is Lemma 2.2 of [29]. The necessity in (b) follows from the theorem. Now suppose that the second sequence is exact. Then $\mathcal{R} \to \mathcal{S}$ is fully faithful by the principle of infinite dévissage (cf., e.g., Lemma 4.2(b) of [18]). We then have to show that the canonical functor $\mathcal{S}/\mathcal{R} \to \mathcal{T}$ is an equivalence. Now by the theorem, we know that $\mathcal{S}^c/\mathcal{R}^c$ identifies with a factor-dense subcategory of $(\mathcal{S}/\mathcal{R})^c$.

By the assumption, $\mathscr{S}^c/\mathscr{R}^c$ also identifies with a factor-dense subcategory of $\mathscr{T}^c$. Hence the functor $(\mathscr{R}/\mathscr{S})^c \to \mathscr{T}^c$ is an equivalence onto a factor-dense subcategory. By the principle of infinite dévissage, it follows that $\mathscr{R}/\mathscr{S} \to \mathscr{T}$ is an equivalence onto a factor-dense subcategory. Since this subcategory has infinite direct sums, it is in fact closed under forming direct summands [2, 3.2] and thus it coincides with $\mathscr{T}$. (c) is the special case where $\mathscr{S}=0$. The last statement is clear because $\mathscr{R}^c$ and $\mathscr{S}^c$ are closed under forming direct summands. □

4.13. Localization

Let

$$0 \to \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \to 0$$

be an exact sequence of $\mathscr{M}_{str}$ such that $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$ are closed DG categories. In the notations of the proof of Proposition 4.1, let $\mathscr{R} \in Cp(\mathscr{A})$, $\mathscr{S} \in Cp(\mathscr{B})$, and $\mathscr{T} \in Cp(\mathscr{C})$ denote $\mathbf{U}$-small subcategories such that $F\mathscr{R} \subset \mathscr{S}$ and $G\mathscr{S} \subset \mathscr{T}$.

Theorem. *The functor G induces a quasi-isomorphism*

$$\mathrm{Cone}(C(\mathscr{R}) \to C(\mathscr{S})) \to C(\mathscr{T}).$$

Remark 1. The proof of the corresponding assertion in [19, Section 6] contained an error: To prove that the functor $\underline{\mathscr{E}} \to \mathscr{H}\mathscr{S}$ of [19, Lemma 5.2] commutes with infinite sums, it is not enough to check that its composition with homology commutes with infinite sums. It is true that there is an equivalence $\underline{\mathscr{E}} \to \mathscr{D}\mathscr{S}$, but probably in general, the functor $\underline{\mathscr{E}} \to \mathscr{H}\mathscr{S}$ does not have its image in $\mathscr{H}_p\mathscr{S}$. To correct the mistake, one needs that $\mathscr{S}$ is a closed DG category. We could not have reduced to this case with the methods at our disposal in [19], but we are now able to do so thanks to the results of Section 3. The mistake is corrected in the proof below.

Proof. We adapt (and correct) the argument of Section 6 of [19]. Consider the sequence

$$0 \to \mathscr{B}(X, FF_\rho Y) \to \mathscr{B}(X, Y) \to \mathscr{B}(X, G_\rho GY) \to 0$$

as a sequence of $\mathscr{S}$–$\mathscr{S}$-bimodules (X and Y denote 'variable' objects of $\mathscr{S}$). We will show that the image of this sequence under the relative left derived functor of $?\otimes_{\mathscr{S}^e} I$, where $I(X,Y)=\mathscr{B}(X,Y)$, is isomorphic to the sequence

$$H(\mathscr{R}) \to H(\mathscr{S}) \to H(\mathscr{T}),$$

where $H(\mathscr{S})$ denotes the Hochschild–Mitchell complex ($=$ b-complex) of $\mathscr{S}$. More precisely, if

$$0 \to L \to M \to N \to 0$$

denotes the bimodule sequence, we will construct a diagram of $\mathscr{S}$–$\mathscr{S}$-bimodules

$$\begin{tikzcd} & L' \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & N' \arrow[d] & \\ 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

whose vertical morphisms are quasi-isomorphisms. It follows that the canonical morphism from the cone over $L' \to M'$ to N' is a quasi-isomorphism. As in the proof of Proposition 4.1, we denote by $\tilde{I}$ a closed resolution of the $\mathscr{S}$–$\mathscr{S}$-bimodule I. We will show that the image of the top row under the functor $? \otimes_{\mathscr{S}^e} \tilde{I}$ is quasi-isomorphic to

$$H(\mathscr{R}) \to H(\mathscr{S}) \to H(\mathscr{T}).$$

It follows that the canonical morphism from the cone over $H(\mathscr{R}) \to H(\mathscr{S})$ to $H(\mathscr{T})$ is a quasi-isomorphism. This implies that the cone over $C(\mathscr{R}) \to C(\mathscr{S})$ is canonically quasi-isomorphic to $C(\mathscr{T})$.

We now construct the resolutions L', M', and N'. For the second term, we take a variation on the bar resolution over $\mathscr{S}$.

For the first term, we take the submodule of the bar resolution over $\mathscr{B}$ which is given by

$$\bigoplus \mathscr{B}(FA_n, FF_\rho Y) \otimes \mathscr{S}(FA_{n-1}, FA_n) \otimes \cdots \otimes \mathscr{S}(FA_0, FA_1) \otimes \mathscr{S}(X, FA_0),$$

where $A_0, \ldots, A_n$ run through $\mathscr{R}$. Since $F\mathscr{R} \subset \mathscr{S}$, this is well defined. We have to show that it is actually a resolution of the first term. For this we have to show that the DG k-module

$$\cdots \to \bigoplus_{A_0} \mathscr{B}(FA_0, FZ) \otimes \mathscr{S}(X, FA_0) \to \mathscr{B}(X, FZ) \to 0$$

is acyclic for each $Z = F_\rho Y$, $Y \in \mathscr{S}$. Indeed, view this DG module as a triangle functor from $\underline{\mathscr{A}}$ to $\mathscr{D}k$, with Z varying in $\underline{\mathscr{A}}$. Clearly, the functor vanishes for $Z \in \mathscr{R}$. Moreover, it commutes with arbitrary direct sums. Thus it vanishes for arbitrary $Z \in \underline{\mathscr{A}}$.

Note that the third term $N' = \mathscr{B}(X, G_\rho GY)$ is isomorphic to $\mathscr{T}(GX, GY)$. We take the bar resolution over $\mathscr{T}$, which we view as an $\mathscr{S}$–$\mathscr{S}$-bimodule via the functor G.

We will now compute the tensor products of L', M', and N' with $\tilde{I}$ over $\mathscr{S}^e$. For M', this amounts to computing

$$(\mathscr{S}(B_n, Y) \otimes O \otimes \mathscr{S}(X, B_0)) \otimes_{\mathscr{S}^e} \tilde{I},$$

where O is a closed DG k-module. Since $\mathscr{S}(B_n, ?)$ and $\mathscr{S}(?, B_0)$ are free, this is quasi-isomorphic to

$$\begin{aligned}(\mathscr{S}(B_n, Y) \otimes O \otimes \mathscr{S}(X, B_0)) \otimes_{\mathscr{S}^e} I &\xrightarrow{\sim} \mathscr{S}(X, B_0) \otimes_{\mathscr{S}} \mathscr{S}(B_n, Y) \otimes O \\ &\xrightarrow{\sim} \mathscr{S}(B_n, B_0) \otimes O\end{aligned}$$

so that we do obtain the b-complex over $\mathscr{S}$.

For L', we have to compute

$$(\mathscr{A}(A_n, F_pY) \otimes O \otimes \mathscr{S}(X, FA_0)) \otimes_{\mathscr{S}^e} \tilde{I}.$$

Since $\mathscr{S}(?, FA_0)$ is free, this is quasi-isomorphic to

$$\begin{aligned}(\mathscr{A}(A_n, F_pY) \otimes O \otimes \mathscr{S}(X, FA_0)) \otimes_{\mathscr{S}^e} I &\xrightarrow{\sim} \mathscr{S}(X, FA_0) \otimes_{\mathscr{S}} \mathscr{A}(A_n, F_pY) \otimes O \\ &\xrightarrow{\sim} \mathscr{A}(A_n, F_pFA_0) \otimes O \\ &\xrightarrow{\sim} \mathscr{R}(FA_n, FA_0) \otimes O,\end{aligned}$$

which is the required result.

Finally, for N', we have to compute

$$(\mathscr{T}(C_n, GY) \otimes O \otimes \mathscr{B}(X, G_pC_0)) \otimes_{\mathscr{S}^e} \tilde{I}.$$

Let $U = \mathscr{C}(C_n, G?)$ viewed as a functor from $\mathscr{B}$ to DG k-modules. Consider the composition

$$\begin{aligned}(U \otimes \mathscr{B}(?, Z)) \otimes_{\mathscr{S}^e} \tilde{I} &\to (U \otimes \mathscr{B}(?, Z)) \otimes_{\mathscr{S}^e} I \\ &\xrightarrow{\sim} \mathscr{B}(?, Z) \otimes_{\mathscr{S}} U \to U(Z)\end{aligned}$$

for $Z \in \mathscr{B}$. Clearly, it is invertible for $Z \in \mathscr{S}$. Moreover, U viewed as a functor $\underline{\mathscr{B}} \to \mathscr{D}k$ commutes with arbitrary coproducts. Thus we have an isomorphism for all $Z \in \underline{\mathscr{B}}$. For $Z = G_pC_0$ we find

$$U(Z) \otimes O = \mathscr{C}(C_n, GG_pC_0) \otimes O \xrightarrow{\sim} \mathscr{T}(C_n, C_0) \otimes O,$$

which is the required result. □

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